# A Variant of the Cramer-Shoup Cryptosystem for Groups of Unknown Order

Stefan Lucks

Theoretische Informatik, Universität Mannheim, 68131 Mannheim, Germany
lucks@th.informatik.uni-mannheim.de

**Abstract.** The Cramer-Shoup cryptosystem for groups of prime order is a practical public-key cryptosystem, provably secure in the standard model under standard assumptions. This paper extends the cryptosystem for groups of unknown order, namely the group of quadratic residues modulo a composed $N$. Two security results are: In the standard model, the scheme is provably secure if both the Decisional Diffie-Hellman assumption for $\mathrm{QR}_N$ *and* the factorisation assumption for $N$ hold. In the random oracle model, the scheme is provably secure under the factorisation assumption by a quite efficient reduction.

## 1 Introduction

Security against chosen ciphertext attacks is essential for many cryptosystems. Naor and Yung [11] introduced this notion into the world of public-key cryptosystems and first described a scheme secure against non-adaptive chosen ciphertext ("lunchtime") attacks. Today, most cryptographers agree that a "good" public-key cryptosystem should be secure against *adaptive chosen ciphertext* (ACC) attacks.[1] This notion has been introduced by Rackoff and Simon [12]. Dolev, Dwork and Naor [9] described a scheme provably secure against ACC attacks under standard assumptions. However, their scheme is too inefficient for practical applications. The research for provably secure and practically efficient cryptosystems has led to schemes provably secure in the *random oracle model* [2], and to schemes provably secure under non-standard assumptions such as the "oracle Diffie-Hellman" assumption [1].

The Cramer-Shoup cryptosystem [5] is the only cryptosystem known to be *both* practical *and* provably secure under standard assumptions – mainly, the decisional Diffie-Hellman assumption in groups of prime order. Recently, the same authors proposed a generalisation of their cryptosystem [7]. Its security can be based either on Paillier's decision composite residiosity assumption or on the (quite classical) quadratic residuosity (QR) assumption – or on the decisional Diffie-Hellman assumption in groups of prime order, as before. As pointed out in

[1] Some authors denote lunchtime attacks by "IND-CCA1" and ACC attacks by "IND-CCA2".

[7], the QR-based variant of the generalisation is not too efficient in practice.[2] In this paper, we deal with another variation, based on the Diffie-Hellman problem in specific groups of non-prime order.

Set $N = PQ$, $P = 2p + 1$, $Q = 2q + 1$, $p \neq q$, and let $P$, $Q$, $p$, and $q$ be odd primes. In the remainder of this paper, we assume $N$ to be of that form. Consider the group $\mathrm{QR}_N$ of the Quadratic Residues mod $N$ and the Cramer-Shoup Cryptosystem in this group. ([5] originally proposed their cryptosystem for groups of prime order only.) As it will turn out, the legal user will not need to know the factorisation of $N$ for either encryption, decryption or key generation (with the possible exception of generating an appropriate $N$ itself). Since knowing the factorisation of $N$ is equivalent to knowing the order of $\mathrm{QR}_N$, the group $\mathrm{QR}_N$ may be *of unknown order* even for the legal user.

A security result in the standard model provides assurance against all attacks, while a random oracle security result only provides assurance against so-called "generic" attacks. On the other hand, it is desirable to base the security of cryptosystems on weak assumptions, instead of strong ones. In this spirit, Shoup [13] proposed a "hedged" variant of the Cramer-Shoup cryptosystem, being both provably secure in the standard model under a strong assumption *and* provably secure in the random oracle model under a weak assumption. In Section 7, we follow the same approach. Our extension is different from Shoup's technique, and the proof for the security in the random oracle model given here is more efficient than its counterpart in [13].

## 2 Properties of the Set $\mathrm{QR}_N$

In this section, we recall some number-theoretic terminology and facts. Let $G$ be a finite multiplicative group of the *order* $|G| \geq 2$. The *order* $\mathrm{ord}(x)$ of $x \in G$ is the smallest integer $e > 0$ such that $x^e = x^0$. $G$ is *cyclic*, if a *generator* $g$ for $G$ exists, i.e., an element $g \in G$ with $\mathrm{ord}(x) = |G|$. Further, $\{1\}$ and $G$ itself are the two *trivial* subgroups of $G$, all other subgroups are nontrivial.

Recall that $N = PQ$, where $P = 2p + 1$, $Q = 2q + 1$, $p$ and $q$ are primes (i.e., both $p$ and $q$ are Sophie-Germain primes). Consider the set $\mathrm{QR}_N = \{x \in \mathbf{Z}_N^* \mid \exists a \in \mathbf{Z}_N^* : a^2 \equiv x \,(\mathrm{mod}\, N)\}$ of Quadratic Residues modulo $N$. In the sequel, we use the following lemmas, which we prove in Section A of the appendix.

**Lemma 1.** *$QR_N$ has a nontrivial subgroup of order $p$ and a nontrivial subgroup of order $q$. Both subgroups are cyclic.*

**Lemma 2.** *$QR_N$ is cyclic. It consists of one element of the order 1, $(p-1)$ elements of the order $p$, $(q-1)$ elements of the order $q$, and $(p-1)(q-1)$ elements of the order $pq$.*

[2] A sample instantiation of the security parameters with $N \approx 2^{1024}$ in [7] implies the following: A public key needs 70 KB of storage space, and an encryption operation needs about 600 exponentiations modulo $N$. Note that the other variants are much more efficient.

**Lemma 3.** *For every* $x \in QR_N$*:* $ord(x) \in \{p, q\} \Rightarrow gcd(x-1, N) \in \{P, Q\}$.

**Lemma 4.** *Let* $g$ *be a generator for* $QR_N$*. For every* $x \in \mathbb{Z}_{pq}$*:* $ord(g^x) \in \{p, q\} \Leftrightarrow gcd(x, pq) \in \{p, q\}$.

Computations in $\mathrm{QR}_N$ are computations modulo $N$. If it is implied by context, we omit writing explicitly "mod $N$" for calculations mod $N$. If $S$ is a finite set, we write $v \in_R S$ if the value $v$ is chosen from the set $S$ according to the uniform probability distribution. We write $x \in_R \mathbb{Z}_{pq}$ for randomly choosing $x$ in $\mathbb{Z}_{pq}$ according to a distribution statistically indistinguishable from uniform. Consider, e.g., $x \in_R \mathbb{Z}_{\lfloor N/4 \rfloor}$. Since $\lfloor N/4 \rfloor \leq pq + p/2 + q/2 + 1/4$, $x \in \mathbb{Z}_{pq}$ is overwhelmingly probable: $\Pr[x \in \mathbb{Z}_{pq}] \geq 1 - \frac{p+q}{2pq} \geq \min\{1 - \frac{1}{p}, 1 - \frac{1}{q}\}$.

## 3 Key Encapsulation Mechanisms

A key encapsulation mechanism (KEM) can be seen as the *secret-key part* of a hybrid cryptosystem. Combining a KEM with an appropriate secret-key cryptosystem provides the functionality of a public-key cryptosystem. If the secret-key cryptosystem satisfies some fairly standard security assumptions and the KEM is secure against ACC attacks, the public-key cryptosystem is secure against ACC attacks as well. (This is called a "folk theorem" in [13]. See also [6].) A KEM is a triple (Gen, KE, KD) of algorithms:

1. A *key pair generation* algorithm Gen, which, given a security parameter, randomly chooses a public-key/secret-key pair (PK,SK).
2. A randomised *key encapsulation* algorithm KE to choose $(C, K)$=KE(PK), i.e. a ciphertext $C$ and an encapsulated key $K$.
3. A deterministic *key decapsulation* algorithm KD to compute $K'$=KD(SK,$C$), and to reject invalid ciphertexts.

A KEM is *sound*, if $K = K'$ for any (PK,SK)=Gen($\cdot$), $(C, K)$=KE(PK), and $K'$=KD(SK,$C$). The KEM presented in Section 4 and its extension in Section 7 are both sound. Proving this is easy, but omitted here for the sake of space.

An *ACC attack* against a KEM (Gen, KE, KD) can be described by the following game:

1. A *key generation* oracle computes (PK,SK)=Gen($\cdot$) and publishes PK.
2. A *key encapsulation* oracle chooses $(C, K)$=KE(PK) and $\sigma \in_R \{0, 1\}$. If $\sigma = 0$, the oracle sends $(C, K)$ to the adversary, else $(C, K')$ with $K' \in_R \{0, 1\}^{|K|}$.
3. The adversary makes some queries $C_1, \ldots, C_q$ to a *key decapsulation* oracle, with $C_i \neq C$. For each query $C_i$, the oracle responds the value KD(SK,$C_i$), which may be either a bit string, or a special code to indicate rejection. For $i \in \{1, \ldots, q-1\}$, the adversary learns the response KD(SK,$C_i$) before she has to choose the next query $C_{i+1}$.
4. The adversary outputs a value $\sigma' \in \{0, 1\}$.

The *adversary's advantage* in guessing $\sigma$ is the difference

$$\left|\mathrm{pr}[\sigma' = 1|\sigma = 1] - \mathrm{pr}[\sigma' = 1|\sigma = 0]\right|$$

of conditional probabilities. A KEM is *secure against ACC attacks*, or *ACC-secure* if, for all efficient adversaries, the advantage is negligible.

In Section B of the appendix, we compare ACC-secure KEMs with ACC-secure public-key cryptosystems and introduce lunchtime-security.

## 4 The Cryptosystem and some Assumptions

Here, we deal with the Cramer-Shoup cryptosystem and what assumptions we make to prove its security. Cramer and Shoup [5] considered groups $G$ of (known) prime order $q^*$, while we consider the group $\mathrm{QR}_N$ of composed order $pq$. There is no need to actually know $pq$, not even for the owner of the secret key. (Note that knowing $pq$ makes factorising $N$ easy.) For the sake of simplicity, we restrict ourselves to describing the system as a key encapsulation mechanism, instead of a full scale public-key cryptosystem.

**Cramer-Shoup cryptosystem in the group $\mathbf{QR}_N$:**

- Key Generation Gen($l$):
  - Generate $N$, $P$, $Q$, $p$, $q$ as above with $2^{l-1} < N < 2^l$.
    Choose a generator $g$ for $\mathrm{QR}_N$.
  - Choose a hash function $H : \{0,1\}^* \to \mathbb{Z}_m$ (with $m \leq pq$).
  - Randomly choose $w \in_R \mathbb{Z}_{pq}$, and compute $g_2 = g^w$. Choose $x_1$, $x_2$, $y_1$, $y_2$, $z \in_R \mathbb{Z}_{pq}$. Compute $c = g^{x_1} g_2^{x_2}$, $d = g^{y_1} g_2^{y_2}$, and $e = g^z$.
  - The public key is PK=($N$, $g$, $H$, $g_2$, $c$, $d$, $e$).
    The secret key is SK=$(x_1, x_2, y_1, y_2, z)$ in $\mathbb{Z}_{pq}^5$.
- Key Encapsulation KE(PK):
  - Choose $r \in_R \mathbb{Z}_{pq}$, compute $u_1 = g^r$, $u_2 = g_2^r$, $k = e^r$, $\alpha = H(u_1, u_2)$ and $t = c^r d^{r\alpha}$.
  - The ciphertext is $(u_1, u_2, t)$, the encapsulated key is $k$.
- Key Decapsulation[3] KD(SK,$(U_1, U_2, T)$) for $(U_1, U_2, T) \in \mathrm{QR}_N^2 \times \mathbb{Z}_N^*$:
  - Compute $K' = U_1^z$, $A' = H(U_1, U_2)$, $T' = U_1^{x_1 + y_1 A'} U_2^{x_2 + y_2 A'}$.
  - If $T = T'$ then output $K'$, else reject.

Both in a group $G$ of prime order and in composed order groups (such as $\mathrm{QR}_N$ and $\mathbb{Z}_N^*$), expressions such as $g^a * g^b$ and $(g^a)^b$ are equivalent to $g^{a+b}$ and $g^{ab}$. For prime order groups "$a + b$" and "$ab$" are addition and multiplication in a *field*, but for general groups $G$ these operations are defined in the *ring* $\mathbb{Z}_{|G|}$. Thus, the proof of security from [5] is not directly applicable to the cryptosystem proposed in the current paper, though our proof is along the same lines.

[3] We don't care if $T \notin \mathrm{QR}_N$, because $T' \in \mathrm{QR}_N$, and the test "$T = T'$" is supposed to fail if $T \notin \mathrm{QR}_N$. Remark 3 describes how to enforce $U_1, U_2 \in \mathrm{QR}_N$.

**Assumption:** (Target collision resistance of $H$)
*Let $F_H$ be a family of hash functions $\{0,1\}^* \to \mathbb{Z}_m$, for $m \leq pq$. Consider the following experiment:*

1. *Fix an input $T$ for $H$ (the "target").*
2. *Randomly choose $H$ from the family $F_H$.*

*It is infeasible to find a "collision" for the target $T$, i.e., an input $T' \neq T$ such that $H(T) = H(T')$.*
As a minor abuse of notation, we write "$H$ is target collision resistant" ("TC-resistant") to indicate that $H$ has been chosen from such a family $F_H$.

**Assumption:** (decisional Diffie-Hellman (DDH) assumption for $\mathrm{QR}_N$)
*Let a generator $g$ for $QR_N$ be given. Consider the distributions $R$ of triples $(g_2, u_1, u_2) \in_R QR_N^3$ and $D$ of triples $(g_2, u_1, u_2)$ with $g_2 \in_R QR_N$, $r \in_R \mathbb{Z}_{pq}$, $u_1 = g^r$, and $u_2 = g_2^r$. It is infeasible to distinguish between $R$ and $D$.* [4]

**Assumption:** (computational Diffie-Hellman (CDH) assumption for $\mathrm{QR}_N$)
*Let a generator $g$ for $QR_N$ be given. Given two values $g_2 \in_R QR_N$ and $u_1 \in_R QR_N$ with $\log_g(u_1) = r$, it is infeasible to find the value $u_2 = g_2^r$.* [5]

**Assumption:** (factoring assumption for $N$)
*Given $N$, it its infeasible to find $P$ or $Q$.*

**Theorem 1 (Factoring assumption ⇒ CDH assumption).**
*If the factoring $N$ is infeasible, the CDH assumption for $QR_N$ holds.*

The proof is in Section C of the appendix.

## 5 Some Technicalities

**Lemma 5.** *Let $g$ be a generator of $QR_N$ and $w \in_R \mathbb{Z}_{pq}$. The value $g_2 = g^w$ is a uniformly distributed random value in $QR_N$. With overwhelming probability, $g_2$ is a generator for $QR_N$.*

*Proof.* Clearly, $g_2$ is uniformly distributed. By Lemma 4, $g_2$ is a generator for $\mathrm{QR}_N \Leftrightarrow w \in \mathbb{Z}_{pq}^*$. Hence, $\mathrm{pr}[g_2 \text{ is a generator for } \mathrm{QR}_N] = (p-1)(q-1)/pq$. □

**Lemma 6.** *If it is feasible to find any pair $(\alpha, \beta) \in \mathbb{Z}_{pq}$ with $(\alpha - \beta) \in \mathbb{Z}_{pq} - \mathbb{Z}_{pq}^* - \{0\}$, it is feasible to factorise $N$.*

[4] An alternative view would be to consider two distributions $D_4$ and $R_4$ of quadruples $(g, g_2, u_1, u_2)$. The distribution of $g$ is the same for $D_4$ and $R_4$, and $g$ is a generator. Apart from that, we don't specify how $g$ is actually chosen. The values $g_2$, $u_1$ and $u_2$ are either chosen according to $D$, or according to $R$.

[5] Since $g \in \mathrm{QR}_N$ is a generator, $\log_g(x)$ is uniquely defined for $x \in \mathrm{QR}_N$.

*Proof.* Let $g$ be a generator for $\mathrm{QR}_N$. If $(\alpha - \beta) \in \mathbb{Z}_{pq} - \mathbb{Z}_{pq}^* - \{0\}$, then $\mathrm{ord}(g^{\alpha-\beta}) \in \{p, q\}$ and thus, we can compute $\gcd(g^{\alpha-\beta} - 1, N) \in \{P, Q\}$. □

**Lemma 7.** *Let $g$ be a generator for $QR_N$ and $g_2 \in_R QR_N$. If it is feasible to choose $u_1, u_2$ such that $u_1 = g^{r_1}$, $u_2 = g_2^{r_2}$, and $(r_2 - r_1) \in \mathbb{Z}_{pq} - \mathbb{Z}_{pq}^* - \{0\}$, it is feasible to factorise $N$.*

*Proof.* Choose $g_2$ as suggested in Lemma 5: $w \in_R \mathbb{Z}_{pq}$; $g_2 = g^w$. Since $(r_2 - r_1) \in \mathbb{Z}_{pq} - \mathbb{Z}_{pq}^* - \{0\}$, $\mathrm{ord}(g^{r_2-r_1}) \in \{p, q\}$. Similarly, $\mathrm{ord}(g_2^{r_2-r_1}) \in \{p, q\}$, and thus $\gcd(g_2^{r_2-r_1}, N) \in \{P, Q\}$. Due to $g_2^{r_2-r_1} = g_2^{r_2}/g_2^{r_1} = u_2/u_1^w$, and since we know $w$, we actually can compute $g_2^{r_2-r_1}$ and thus factorise $N$. □

Now we describe a simulator for the Cramer-Shoup cryptosystem. Its purpose is not to be actually used for key encapsulation and decapsulation, but as a technical tool for the proof of security. If an adversary mounts an attack against the Cramer-Shoup cryptosystem, the simulator may provide the responses, instead of an "honest" Cramer-Shoup oracle. Note that the adversary can make many key decapsulation queries, but only one single key encapsulation query.

**A Simulator for the Cramer-Shoup Cryptosystem in $\mathbf{QR}_N$:**

- Generate the public key:
  - Let the values $g$, $N$ and $H$ and a triple $(g_2, u_1, u_2) \in \mathrm{QR}_N^3$ be given.
  - Choose $x_1, x_2, y_1, y_2, z_1, z_2 \in_R \mathbb{Z}_{pq}$. Compute $c = g^{x_1} g_2^{x_2}$, $d = g^{y_1} g_2^{y_2}$, and $e = g^{z_1} g_2^{z_2}$. [6] The public key is PK=($N$, $g$, $H$, $g_2$, $c$, $d$, $e$).
- Key Encapsulation KE(PK):
  - Compute $k = u_1^{z_1} u_2^{z_2}$, $\alpha = H(u_1, u_2)$, $t = u_1^{x_1+y_1\alpha} u_2^{x_2+y_2\alpha}$.
  - The ciphertext is $(u_1, u_2, t)$, the encapsulated key is $k$.
- Key Decapsulation KD(SK,$(U_1, U_2, T)$):
  - Compute $K' = U_1^{z_1} U_2^{z_2}$, $A' = H(U_1, U_2)$, $T' = U_1^{x_1+y_1 A'} U_2^{x_2+y_2 A'}$.
  - If $T = T'$ then output $K'$, else reject.

## 6 A Proof of Security in the Standard Model

In this section, we prove the security of the Cramer-Shoup Cryptosystem in $\mathrm{QR}_N$ in the standard model. The proof is based on three lemmas.

**Theorem 2 (Security in the standard model).**
*If $H$ is TC-resistant and both the DDH assumption for $QR_N$ and the factoring assumption for $N$ hold, the Cramer-Shoup cryptosystem in $QR_N$ is ACC-secure.*

**Lemma 8.** *If the triple $(g_2, u_1, u_2)$ given to the simulator is distributed according to distribution $D$, an adversary cannot statistically distinguish between the behavior of the simulator and the Cramer-Shoup cryptosystem itself.*

[6] In contrast to the simulator, the cryptosystem itself implicitly defines $z_2 = 0$.

*Proof.* If $(g_2, u_1, u_2)$ is distributed according to $D$, a value $r$ exists such that $u_1 = g^r$ and $u_2 = g_2^r$. We show that the simulator's responses are statistically indistinguishable from the *real* cryptosystem's responses.

Consider the key encapsulation query. The simulator computes

$$k = u_1^{z_1} u_2^{z_2} = g^{rz_1} g_2^{rz_2} = (g^{z_1} g_2^{z_2})^r = e^r,$$
$$\alpha = H(u_1, u_2) \text{ and}$$
$$t = u_1^{x_1+y_1\alpha} u_2^{x_2+y_2\alpha} = g^{rx_1+ry_1\alpha} g_2^{rx_2+ry_2\alpha} = g^{rx_1} g_2^{rx_2} g^{ry_1\alpha} g_2^{ry_2\alpha} = c^r d^{r\alpha}.$$

The distribution of the response $((g_2, u_1, u_2), k)$ is identical to the distribution of the cryptosystem's response.

Now consider the key decapsulation queries. If a query $(U_1, U_2, T)$ is valid, i.e., if a value $R \in \mathbb{Z}_{pq}$ exists with $U_1 = g^R$ and $U_2 = g_2^R$, the simulator's response is the same as the response the cryptosystem provides. Both the simulator and the cryptosystem reject $(U_1, U_2, T)$ if $T \neq T' = U_1^{x_1+y_1A'} U_2^{x_2+y_2A'}$, and else output $K' = U_1^{z_1} U_2^{z_2} = (g^R)^{z_1} (g_2^R)^{z_2} = (g^{z_1} g_2^{z_2})^R = e^R$. It remains to show that both the cryptosystem and the simulator (given $(g_2, u_1, u_2)$ distributed according to $D$) reject all invalid key decapsulation queries with overwhelming probability – and thus essentially behave identically.

The decision to reject an invalid ciphertext $(U_1, U_2, T)$ depends on four random values $x_1, x_2, y_1, y_2 \in \mathbb{Z}_{pq}$. A part of the public key are the values $c$ and $d$ with $c = g^{x_1} g_2^{x_2} = g^{x_1} g^{wx_2}$ and $d = g^{y_1} g_2^{y_2} = g^{y_1} g^{wy_2}$, i.e.,

$$l_c := \log_g(c) = x_1 + wx_2 \iff x_1 = l_c - wx_2 \text{ and} \tag{1}$$
$$l_d := \log_g(d) = y_1 + wy_2. \iff y_1 = l_d - wy_2 \tag{2}$$

These equations[7] provide public information about the quadruple $(x_1, x_2, y_1, y_2)$ of secret values. The response to the encapsulation query provides another equation $\log_g(t) = rx_1 + ry_1\alpha + rwx_2 + rwy_2\alpha$, however $\log_g(t) = rl_c + rl_d\alpha$, i.e., this new equation linearly depends on Equations 1 and 2, and thus provides no new information about $(x_1, x_2, y_1, y_2)$. This still leaves $(pq)^2$ possibilities for the quadruple $(x_1, x_2, y_1, y_2)$.

Assume $g_2$ to be a generator for $\mathrm{QR}_N$. (By Lemma 5, this is overwhelmingly probable.) Let the ciphertext $(U_1, U_2, T)$ be invalid. Thus, $R_1 \neq R_2$ exist with $U_1 = g^{R_1}$ and $U_2 = g_2^{R_2}$. To answer the query, the values $K' = U_1^{z_1} U_2^{z_2}$ (or $K' = U_1^z$), $A' = H(U_1, U_2)$, and $T' = U_1^{x_1+y_1A'} U_2^{x_2+y_2A'} = g^{R_1x_1+R_1y_1A'} g_2^{R_2x_2+R_2y_2A'}$ are computed, which provides the equation

$$l_{T'} := \log_g(T') = R_1x_1 + R_1y_1A' + wR_2x_2 + wR_2y_2A'. \tag{3}$$

Equations 1 and 2 can be used to eliminate the variables $x_1$ and $y_1$:

$$\begin{aligned} l_{T'} &= R_1l_c - R_1wx_2 + R_1l_dA' - R_1wy_2A' + wR_2x_2 + wR_2y_2A' \\ &= R_1l_c + R_1l_dA' + wx_2(R_2 - R_1) + wy_2A'(R_2 - R_1) \end{aligned}$$

[7] It is vital that $l_c$ and $l_d$ are uniquely defined. We need not actually compute $l_c$ or $l_d$.

By Lemma 5 and Lemma 7 we know that with overwhelming probability and under the factoring assumption both $w$ and $(R_2 - R_1)$ are invertible mod $pq$. If these two values are invertible, we may fix the value $y_2$ arbitrarily and there always exists a uniquely defined value

$$x_2 = \frac{l_{T'} - R_1 l_c - R_1 A' l_d - w y_2 A'(R_2 - R_1)}{w(R_2 - R_1)}$$

to prevent the rejection of the invalid ciphertext $(U_1, U_2, T)$. Each time an invalid ciphertext is rejected, this eliminates at most $pq$ of the $(pq)^2$ possible quadruples $(x_1, x_2, y_1, y_2)$. □

**Lemma 9.** *If the triple $(g_2, u_1, u_2) \in QR_N^3$ given to the simulator is distributed according to distribution $R$, the simulator rejects all invalid ciphertexts with overwhelming probability.*

*Proof.* Recall that the rejection of an invalid ciphertext $(U_1, U_2, T)$ depends on the quadruple $(x_1, x_2, y_1, y_2) \in \mathrm{QR}_N$ of secret values, and that the public key provides the two linear Equations 1 and 2 to narrow down the number of possibilities for $(x_1, x_2, y_1, y_2)$ to $(pq)^2$. The response to the encapsulation query provides the value $t = u_1^{x_1 + y_1\alpha} u_2^{x_2 + y_2\alpha}$ and thus a linear equation

$$l_t := \log_g(t) = r_1 x_1 + r_1 y_1 \alpha + w r_2 x_2 + w r_2 y_2 \alpha. \tag{4}$$

By using Equations 1 and 2, we can eliminate the variables $x_1$ and $y_1$:

$$\begin{aligned} l_t &= r_1 l_c - r_1 w x_2 + r_1 l_d \alpha - r_1 w x_2 \alpha + w r_2 x_2 + w r_2 y_2 \alpha \\ &= r_1 l_c + r_1 l_d \alpha + w x_2 (r_2 - r_1) + w y_2 \alpha (r_2 - r_1) \\ \Rightarrow y_2 &= \frac{l_t - r_1 l_c - r_1 l_d - w x_2 (r_2 - r_1)}{w(r_2 - r_1)\alpha} \end{aligned}$$

An invalid ciphertext $(U_1, U_2, T)$ is rejected, except when Equation 3 holds, which means $T' = T$. Recall $\alpha = H(u_1, u_2)$ and $A' = H(U_1, U_2)$ and consider three cases:

- Case 1, $(U_1, U_2) = (u_1, u_2)$: By the definition of an ACC attack, we require $t \neq T$, and thus the key decapsulation query $(U_1, U_2, T)$ will be rejected.
- Case 2, $(U_1, U_2) \neq (u_1, u_2)$ and $\alpha = A'$: This is a collision for $H$ for the target $(u_1, u_2)$, which contradicts the assumption for $H$ to be TC-resistant.
- Case 3: $(U_1, U_2) \neq (u_1, u_2)$ and $\alpha \neq A'$: We have four unknowns $x_1, x_2, y_1, y_2 \in \mathrm{QR}_N$, and four Equations 1, 2, 3, and 4 describe their relationship. By solving this system of linear equations we get

  $$y_2 = \frac{l_{T'} - r_1 l_c - \frac{l_t - r_2 l_c - r_1 l_d \alpha}{r_2 - r_1}(R_2 - R_1) - A' R_1 l_d}{(R_2 - R_1) w (A' - \alpha)},$$

  which uniquely determines $y_2$ if all the four values $(r_2 - r_1)$, $(R_2 - R_1)$, $w$, and $(A' - \alpha)$ are invertible in $\mathbb{Z}_{pq}$.[8] The invertibility of $(r_2 - r_1)$ and $(R_2 - R_1)$ follows from Lemma 7, the invertibility of $w$ follows from Lemma 5, and the invertibility of $(A' - \alpha)$ follows from Lemma 6. □

[8] This implies that the four linear equations 1, 2, 3, and 4 are linearly independent.

**Lemma 10.** *Let $k$ be the encapsulated key in the response for the encapsulation query. If the triple $(g_2, u_1, u_2) \in QR_N^3$ given to the simulator is distributed according to distribution $R$, it is infeasible for the adversary to distinguish between $k$ and a uniformly distributed random value.*

*Proof.* We set $r_1 = \log_g(u_1)$ and $r_2 = \log_{g_2}(u_2)$. Assume that $g_2 = g^w$ is a generator for $\mathrm{QR}_N$ and that $r_1 \neq r_2$. Both assumptions hold with overwhelming probability. Now we prove: *If all invalid decapsulation queries are rejected during the simulation, then under the factoring assumption it is infeasible for the adversary to distinguish between $k$ and a random value.*

Observe that $k$ only depends on the two random values $z_1, z_2 \in \mathrm{QR}_N$. Since $e = g^{z_1} g_2^{z_2}$, the public key provides one linear equation

$$l_e := \log_g(e) = z_1 + wz_2 \iff z_1 = l_e - wz_2. \tag{5}$$

The rejection of an invalid key decapsulation query does not depend on $z_1$ and $z_2$. If the decapsulation query $(U_1, U_2, T)$ is valid and not rejected, we have a value $R$ such that $U_1 = g^R$ and $U_2 = g_2^R$. By $\log_g(k) = Rz_1 + Rwz_2 = R\log_g(e)$ this provides another equation, linearly depending on Equation 5. The response for the key encapsulation query consists of a ciphertext $(u_1, u_2, t)$ and a key $k = u_1^{z_1} u_2^{z_2} = g^{r_1 z_1} g^{w r_2 z_2}$, which provides a linear equation

$$l_k := \log_g(k) = r_1 z_1 + w r_2 z_2 = r_1 l_e - r_1 w z_2 + r_2 w z_2 = r_1 l_e + w z_2 (r_2 - r_1), \tag{6}$$

which finally gives

$$z_2 = \frac{l_k - r_1 l_e}{w(r_2 - r_1)}.$$

As before, we argue that with overwhelming probability and under the factoring assumption both $w$ and $(r_2 - r_1)$ are invertible in $\mathbb{Z}_{pq}$. If $w$ and $(r_2 - r_1)$ are invertible, then a unique value $z_2$ exists for every key $k \in \mathrm{QR}_N$. □

*Proof (Theorem 2).* If the adversary can break the cryptosystem by distinguishing a *real* encapsulated key from a *random* one, she can do so as well in the simulation, if the simulator input is chosen according to distribution $D$ (Lemma 8). Since she cannot distinguish a real key from a random key in the simulation if the simulator input is distributed according to $R$ (Lemmas 9 and 10), being able to break the cryptosystem means being able to distinguish distribution $D$ from distribution $R$, contradicting the DDH assumption for $\mathrm{QR}_N$. □

*Remark 1 (Strengthening Theorem 2 by avoiding the factoring assumption).* If $H$ is TC-resistant and the DDH assumption for $\mathrm{QR}_N$ holds, the Cramer-Shoup Cryptosystem in $\mathrm{QR}_N$ is ACC-secure.

To verify this, assume that the adversary somehow learns the factors $P$ and $Q$ of $N$. Then the DDH-problem for $\mathrm{QR}_N$ is hard if and only if both the DDH problem for $\mathrm{QR}_P$ and the DDH problem for $\mathrm{QR}_Q$ are hard. But given $P$ and $Q$ and an oracle to mount an ACC-attack against the Cramer-Shoup Cryptosystem for $\mathrm{QR}_N$, we can use this oracle to solve the DDH problem for either $\mathrm{QR}_P$ or $\mathrm{QR}_Q$. In this case, the DDH problem for $\mathrm{QR}_N$ is feasible.

## 7 An Extension and its Security

We describe how to extend the Cramer-Shoup cryptosystem, dealing with a hash function $h$, which may be used like a random oracle ($\rightarrow$ Figure 1):

### 7.1 The Extended Scheme and its Abstract Security

**Cramer-Shoup cryptosystem in $\mathrm{QR}_N$ with $h$-extension:**

- The key pair (PK, SK) is the same as for the non-extended Cramer-Shoup cryptosystem. Let $h$ be a function $h : \{1, 2, 3\} \times \mathrm{QR}_N^3 \rightarrow \mathrm{QR}_N$.
- Extend key encapsulation by computing
  $t_* = t * h(1, k, u_1, u_2)$ ($\rightarrow$ solid arrows in Figure 1) and
  $\tau = t * h(2, k, u_1, u_2)$ and $k_* = k * h(3, \tau, u_1, u_2)$ ($\rightarrow$ dashed arrows).
  The ciphertext is $(u_1, u_2, t_*)$, the encapsulated key is $k_*$.
- Decapsulate the ciphertext $(U_1, U_2, T_*) \in \mathrm{QR}_N^2 \times \mathbb{Z}_N^*$ by computing $K'$, $T'$ as before, and reject if $T' * h(1, K', U_1, U_2) \neq T_*$. Else compute $\tau' = T' * h(2, K', U_1, U_2)$ and output $K'_* = K' * h(3, \tau', U_1, U_2)$.

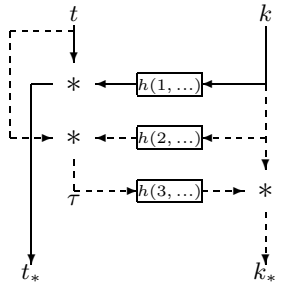


**Fig. 1.** The $h$-extension: converting $t$ and $k$ into $t_*$ and $k_*$.

**Theorem 3 (Security of $h$-extended scheme in standard model).** *Let $h$ be any efficient function $h : \{1, 2, 3\} \times QR_N^3 \rightarrow QR_N$. If $H$ is TC-resistant and both the DDH assumption for $QR_N$ and the factoring assumption for $N$ hold, the Cramer-Shoup cryptosystem in $QR_N$ is ACC-secure.*

*Proof.* Observe that the simulator described in Section 5 computes the values $k$ and $t$ when dealing with an encapsulation query. Also, being asked to decapsulate the ciphertext $(U_1, U_2, T)$, the same simulator computes the values $K'$ and $T'$ from $U_1$ and $U_2$. Thus, it is straightforward to apply the $h$-extension to the simulator. Since $h$ is efficient, the extended simulator is efficient, too.

Using the extended simulator instead of the original one, the proof for Theorem 2 is applicable to Theorem 3. □

**Theorem 4 (Security of $h$-extended scheme in random oracle model).** *If the function $h$ is modeled as a random oracle, the $h$-extended scheme is ACC-secure under the factoring assumption.*

*Proof.* Let $N$ and $H$ be given. Consider an adversary with a non-negligible advantage to win the attack game. In the following experiment, we modify the key generation and we describe how to respond to the adversary's oracle queries, including queries to the random oracle. We start with the key generation:

- Choose $\beta \in_R \{1, \ldots, \lfloor N/4 \rfloor - 1\}$, $\alpha \in_R \mathbb{Z}_N^*$ and compute $e := \alpha^2$.
- Choose $u_1 \in_R \mathrm{QR}_N$ and compute $g := u_1^{2\beta}$. (We will search for $k = e^{\log_g(u_1)}$, i.e., for the value $k$ with $k^{2\beta} = e$. If we find $k$, we have a 50 % chance that $\gcd(k^\beta - \alpha, N) \in \{P, Q\}$ holds, providing us with the factorisation of $N$.)
- Choose $w \in_R \mathbb{Z}_{pq}$ and compute $g_2 = g^w$ and $u_2 = u_1^w$.
- Choose $x_1, x_2, y_1, y_2 \in_R \mathbb{Z}_{pq}$ and compute $c = g^{x_1} g_2^{x_2}$ and $d = g^{y_1} g_2^{y_2}$.
- Use $(N, g, H, g_2, c, d, e)$ as the public key.

The response to the key encapsulation query is the ciphertext $(u_1, u_2, t_*)$ and the encapsulated key $k_*$ with $t_*, k_* \in_R \mathrm{QR}_N$.

Let $(U_1, U_2, T_*)$ be a key decapsulation query. We respond as follows:

- Compute $T' = U_1^{x_1 + y_1 A'} U_2^{x_2 + y_2 A'}$.
- Consider values $K'$ with queries for $\delta_1 = h(1, K', U_1, U_2)$ to the random oracle. Verify, if for one such value $K'$ the equation

$$(K')^{2\beta} = U_1 \tag{7}$$

  holds. If not, or if $T_* \neq T' * \delta_1$, then reject.
- Else ask the random oracle for $\delta_2 = h(2, K', U_1, U_2)$, compute $\tau = T' * \delta_2$, ask for $\delta_3 = h(3, \tau, U_1, U_2)$, and respond $K'_* = K' * \delta_3$ to the adversary.

A random oracle query to compute $h(I, X, U_1, U_2)$ (with $I \in \{1, 2, 3\}$ and $X$, $U_1$, $U_2 \in \mathrm{QR}_N$) may be asked either by the adversary, or by ourselves when answering a key decapsulation query. The answer is computed as follows:

1. If we have been asked for $h(I, X, U_1, U_2)$ before, repeat the same answer.
2. Else, if $I \in \{1, 2\}$, $u_1 = U_1$, $u_2 = U_2$, and $X^{2\beta} = e$, print $X$ and abort.
3. Else choose $Y \in_R \mathrm{QR}_N$ and respond $Y$.

Observe that if we never abort ($\rightarrow$ Step 2), the adversary cannot distinguish $h$ from a random function over the same domain. On the other hand, assume that we abort the experiment, having found a value $X$ with $X^{2\beta} = e$, i.e., a square root (mod $N$) of $e$. Initially, we know two square roots of $e$, namely $\pm\alpha$. Since the adversary has no information about $\alpha$, except for $e = \alpha^2$, $X^\beta \neq \pm\alpha$ holds with probability 1/2. In this case, we can factorise $N$ by computing $\gcd(X^\beta - \alpha, N) \in \{P, Q\}$. This shows: *If $\pi$ is the probability to abort the experiment, we can factorise $N$ with the probability $\pi/2$ after running the experiment once.*

Now, we deal with three different games:

1. The attack game with the "real" encapsulated key $k_*$,
2. the attack game where $k_*$ is replaced by a random value, and
3. the experiment we defined for the current proof.

As it turns out, the adversary cannot distinguish the experiment from either of the attack games, except when we abort the experiment:

- The public key values $g$, $g_2$, $c$, $d$, and $e$ are independent uniformly distributed random values in $\mathrm{QR}_N$ – in the attack games, as in the experiment.
- In the attack games, the values $u_1$ and $u_2$ from the encapsulation query satisfy the equation $u_2 = g_2^{\log_g(u_1)}$, with $u_1 \in_{\mathrm{R}} \mathrm{QR}_N$. For one of the attack games, the values $t_*$ and $k_*$ depend on $t$ and $k$ (and $h$), while for the other one, $t_*$ depends on $t$ and $k$, while $k_*$ is chosen at random.
  In the experiment, $u_1 \in_{\mathrm{R}} \mathrm{QR}_N$ and $u_2 = g_2^{\log_g(u_1)}$ as well. The value $t_*$ cannot be distinguished from a uniformly distributed random value without asking for $h(1, k, u_1, u_2)$ (and then aborting). The value $k_*$ cannot be distinguished without asking for $h(3, \tau, u_1, u_2)$. Asking this query is infeasible without having asked for $\delta_2 = h(2, k, u_1, u_2)$ (followed by an abortion), since $\tau$ depends on $\delta_2$.
- Consider a decapsulation query $(U_1, U_2, T_*)$. Let $K'$ be defined by Equation 7. If $h$ is a well-defined function, there is a unique well defined value $T'_*$ such that a ciphertext $(U_1, U_2, T'_*)$ has to be accepted, and every ciphertext $(U_1, U_2, T_*)$ with $T_* \neq T'_*$ has to be rejected. Without asking for $h(1, K', U_1, U_2)$, the adversary cannot predict $T'_*$, and any ciphertext $(U_1, U_2, T_*)$ chosen by the adversary is rejected with overwhelming probability in the attack games and with probability 1 in the experiment.
  If the adversary had asked for $h(1, K', U_1, U_2)$, the computation of $T'_*$ and $K'_*$ is exactly the same in the experiment as in the attack games. □

### 7.2 The Concrete Security of the Extended Scheme

Note that the reduction in the proof of Theorem 4 is very efficient. We quantify this by describing the *concrete security* against a generic adversary, i.e., against an adversary who treats the hash function $h$ like a random oracle.

**Theorem 5 (Concrete security of $h$-extended scheme in r. o. model).** *Let $\mathcal{A}$ be a generic ACC adversary, allowed to ask one key encapsulation query, $q_{\mathrm{KD}}$ key decapsulation queries, and $q_1 + q_2 + q_3$ random oracle queries, namely $q_i$ random oracle queries of the form $h(i, \ldots)$. Assume $\mathcal{A}$ takes the running time $T_{\mathcal{A}}$ and achieves the advantage $a_{\mathcal{A}}$ when distinguishing between the attack game with the "real" and the attack game with a random encapsulated key.*
*Then an algorithm $\mathcal{F}$ exists to find the factors $P$ and $Q$ of $N$ with at least the probability $a_{\mathcal{A}}/2 - (q_3 + 2q_{\mathrm{KD}})/pq$. The expected running time for $\mathcal{F}$ is at most $T_{\mathcal{A}} + T_\delta$, with $T_\delta$ being linear in the total number $q_\Sigma = (1 + q_{\mathrm{KD}} + q_1 + q_2 + q_3)$ of oracle queries. More specifically, $T_\delta$ is the time for doing $7 + 3q_{\mathrm{KD}} + q_1 + q_2$ exponentiations mod $N$ and $O(q_\Sigma)$ other operations.*

*Proof.* The proof of Theorem 4 already describes what we call algorithm $\mathcal{F}$, here: *Run the key generation and then invoke the distinguishing adversary $\mathcal{A}$, providing all responses to $\mathcal{A}$'s oracle queries.* To prove Theorem 5, we concretely analyse running time and probability of success of this algorithm.

Running time:

During key generation, we compute seven values by exponentiation mod $N$: $u_1^{2\beta}$, $g^w$, $u_1^w$, $g^{x_1}$, $g_2^{x_2}$, $g^{x_1}$, $g_2^{y_2}$. When responding to the key encapsulation query, no exponentiations are necessary. Responding to a random oracle query $h(1,\ldots)$ or $h(2,\ldots)$ may require to compute $X^{2\beta}$. Queries $h(3,\ldots)$ can be answered without computing any exponentiations.

Key decapsulation queries are slightly more complicated and may need up to three exponentiations. Two are needed to to compute $T'$. The values $(K')^{2\beta}$ have already been computed when dealing with a random oracle query $h(1,\ldots)$ and may have been stored in a table. But responding to a key decapsulation query may require to make two additional calls $h(2,\ldots)$ and $h(3,\ldots)$ to the random oracle, and calling $h(2,\ldots)$ may require another (third) exponentiation.

Thus the total number of exponentiations mod $N$ is at most

$$3q_{\mathrm{KD}} + q_1 + q_2 + 7.$$

Similarly, we can count the total number of other operations, which is no more than linear in $q_\Sigma$, as well.

Note that the random oracle may have to respond to at most $q_1$ queries $h(1,\ldots)$, but to at most $q_{\mathrm{KD}} + q_2$ queries $h(2,\ldots)$ and $q_{\mathrm{KD}} + q_3$ queries $h(3,\ldots)$. The reason is, that computing the answer to a decapsulation query may include two additional random oracle queries $h(2,\ldots)$ and $h(3,\ldots)$.

Probability:

$\mathcal{A}$ cannot distinguish between the two attack games without asking for $h(3,\tau,u_1,u_2)$, where $\tau = h(2,k,u_1,u_2) * t$. If $\mathcal{A}$ ever asks for $h(2,k,u_1,u_2)$, the simulator aborts and $\mathcal{F}$ will factorise $N$ with a 50 % probability of success. Else, $\mathcal{A}$ has no (Shannon-) information about $\tau$. In this case, and since at most $q_3 + q_{\mathrm{KD}}$ queries of the form $h(3,\ldots)$ are to be answered, the probability that any of these is of the form $h(3,\tau,u_1,u_2)$ is at most $(q_3 + q_{\mathrm{KD}})/pq$.

When might the adversary be able to distinguish either of the attack games from the experiment we define in the proof of Theorem 4, i.e., from the behavior of algorithm $\mathcal{F}$? The experiment behaves exactly like any of the attack games, with the following two exceptions:

1. $\mathcal{A}$ asks for $h(I,X,U_1,U_2)$ with $I \in \{1,2\}$ and $X^{2\beta} = e$. In this case, the experiment is aborted (and $\mathcal{F}$ has a 50 % chance of factorising $N$).
2. $\mathcal{A}$ asks for the decapsulation of a ciphertext $(U_1,U_2,T_*)$, without having asked for $h(1,K',U_1,U_2)$ before ($K'$ is defined in Equation (7)). In this case, $\mathcal{F}$ always rejects, while the attack games reject with the probability $1/pq$.

Since $\mathcal{A}$ can ask for the decapsulation of $q_{\mathrm{KD}}$ ciphertext, the entire probability that any random ciphertext is not rejected in an attack game is $\leq q_{\mathrm{KD}}/pq$.

Even if $\mathcal{A}$ could always distinguish the "real" encapsulated key from a random value when asking for $h(3,\tau,u_1,u_2)$ without asking for $h(2,k,u_1,u_2)$ before, or when a random ciphertext $(U_1,U_2,T_*)$ in a key decapsulation query is not

rejected, the probability for $\mathcal{F}$ to factorise $N$ would not be less than

$$\frac{a_{\mathcal{A}}}{2} - \frac{q_3 + q_{\mathrm{KD}}}{pq} - \frac{q_{\mathrm{KD}}}{pq}.$$

□

*Remark 2 (Practical consequences of Theorem 5).*
Theorem 5 counts modular exponentiations and mentions "other operations". These are simpler arithmetic operations (e.g. multiplication mod $N$), choosing random values ($\in_{\mathrm{R}}$ $\mathrm{QR}_N$, $\in_{\mathrm{R}}$ $\mathbb{Z}_N^*$, and $\in_{\mathrm{R}}$ $\mathbb{Z}_{pq}$), and hash table look-up and update operations. In practice, none of these operations should be slower than an exponentiation mod $N$. Thus, the running time for algorithm $\mathcal{F}$ is $T_{\mathcal{A}}+O(q_\Sigma * T_N)$, where $T_N$ is the time for computing an exponentiation mod $N$.
For any reasonable choice of $N$, the probability that $F$ actually factorises $N$ is extremely close to $a_{\mathcal{A}}/2$.

### 7.3 Comparison to Shoup's technique

The approach in this section has been inspired by Shoup [13], who also described a "hedged" variant of the Cramer-Shoup Scheme, being both

- provably secure in the standard model under a "strong" assumption *and*
- provably secure in the random oracle model under a "weak" assumption.

In [13], the "strong" assumption is the DDH assumption for a group $G$ of prime order. The "weak" assumption is the CDH assumption for $G$. As was stressed in [6] (see also [13, Remark 4]), the reduction in the random oracle model is quite inefficient, since it is relative to a DDH oracle:

- Let the DDH assumption for $G$ be false. I.e., a polynomial-time algorithm $A_1$ with a *significant* DDH advantage exists. By standard amplification techniques (calling $A_1$ polynomially often), we get $A_2$, which achieves an overwhelming DDH advantage. Note that the DDH-oracle $A_2$ is "efficient" in the sense of Complexity Theory, but may be quite inefficient in practice.
  Assume an efficient generic ACC adversary $A$ exists to break the hedged Cramer-Shoup variant [13]. The reduction in [13] describes how to use the adversary $A$ as a tool to break the CDH assumption for $G$. The reduction requires to call $A_2$ each time when $A$ asks a new random oracle query.
- Consider a hypothetical example (using viciously chosen numbers):
  *Let, for some choice of $G$, $A_1$ run in $2^{30}$ computer clocks. Thus, $A_1$ could qualify as "practically efficient". If $A_2$ executes $A_1$ $2^{30}$ times, $A_2$ could be considered "hard, but feasible" on a massively parallel computer. Now consider an efficient generic ACC adversary $A$ making $2^{30}$ random oracle queries. The reduction provides an algorithm to solve the CDH problem for $G$, but this algorithm would require more than $2^{90}$ units of time.*
  Thus, an efficient generic ACC attack against the scheme does not necessarily reduce to a practical solution for the CDH problem for $G$.

As explained above, the reduction in the current paper is quite efficient, using linearly many *moderately simple* operations (such as exponentiations mod $N$), but no *potentially complex* operation (such as the DDH oracle in [13]).

Also note that we do not assume the hash function $H$ to be TC-resistant, for Theorem 4, in contrast to [13, Theorem 3].

On the other hand, the random oracle security in the current paper is based on the factoring assumption, not on the CDH assumption. This may be seen as a disadvantage. By generalising the technique from [13] for $\mathrm{QR}_N$, we might be able to use the CDH assumption for $\mathrm{QR}_N$ instead, which is at least as strong as the factoring assumption for $N$, see Theorem 1.

A rather technical difference to our approach is that [13] introduces the notion of a *pair-wise independent hash function* (PIH) and combines a PIH with a random oracle. The PIH is required for the security result in the standard model (i.e., for the counterpart of Theorem 3 in the current paper).

## 8 Final Remarks and Discussion

*Remark 3 (The input for KD).*
Note that the input $(U_1, U_2, T^*) \in \mathrm{QR}_N^2 \times \mathbb{Z}_N^*$ is under control of the adversary. If $x$ is a number, it is easy to verify whether $x \in \mathbb{Z}_N^*$, but it may be difficult to verify $x \in \mathrm{QR}_N$. We can deal with this problem by using KE' and KD' instead of KE and KD:

- KE': Compute KE and replace $k$ by $k^2$ and $t$ by $t^2$.
- KD'(SK,$(U_1, U_2, T)$) for $(U_1, U_2, T) \in (\mathbb{Z}_N^*)^3$: Compute KD(SK,$(U_1^2, U_2^2, T)$).

Note that (Gen,KE',KD') is as sound as (Gen,KE,KD). But for $(U_1, U_2, T) \in (\mathbb{Z}_N^*)^3$, the input for $KD$ is now in $\mathrm{QR}_N^2 \times \mathbb{Z}_N^*$, as it should. A similar technique can be used for the $h$-extension.

*Remark 4 (The hash function H).*
Theoretically we don't need an *additional* assumption for the TC-resistance of $H$. Based on the factoring assumption, provably secure TC-resistant (and even stronger) hash functions are known. In practice, we may prefer to use a dedicated hash function such as SHA-1 or RIPE-MD 160.

Recall that we deal with a cryptosystem which has computations in $\mathrm{QR}_N$, but nobody knows (or needs to know) the order of $\mathrm{QR}_N$. (Note that knowing the order of $\mathrm{QR}_N$ is equivalent to knowing the factors of $N$.)

This may be interesting for the construction of advanced cryptographic protocols, where some party knows the factorisation of $N$ *in addition* to the secret key, while another party only knows the secret key itself. E.g., consider a variant of our scheme, where the factors of $N$ (possibly more than just two, in contrast to the current paper) are small enough that computing the discrete log modulo any of the factors is feasible. Everyone knowing the factorisation of $N$ can thus compute discrete logarithms mod $N$, and the factorisation of $N$ may serve

as a "master key": Knowing it allows to compute the secret key from a given public key defined over the group $\mathrm{QR}_N$. This approach is roughly related to key insulation [8], where "ordinary" public keys may be stored and used in a risky environment, while the master key is well protected.

From a practical point of view, it may not appear too useful to hide the order of the group from the owner of the secret key (except in the context of the advanced protocols mentioned above). In practice, the owner of the secret key might want to use the knowledge of the factors $P$ and $Q$ of $N$ to improve the efficiency of key decapsulation by applying the Chinese Remaindering Theorem.

The main practical selling point for the current scheme is the improved security assurance in the random oracle model, compared to [13].

An interesting open problem is the following: *Is this paper's hedging technique (cf. Figure 1) applicable to other cryptosystems, e.g., to the variants of the Cramer-Shoup Cryptosystem described in [7]?*

## Acknowledgement

Eike Kiltz [10] first observed Remark 1. He and the Asiacrypt referees provided many hints to improve this presentation. Ronald Cramer [4] pointed out that the framework from [7] can be used for an alternative proof of Theorem 2 and Remark 1. Following a referee's advice to give more attention to concrete security, Theorem 5 has been added to this final version of the paper.

## References

1. M. Abdalla, M. Bellare, P. Rogaway: "DHAES: an encryption scheme based on the Diffie-Hellman problem", preprint, March 17, 1999, `http://eprint.iacr.org/1999/007/`.
2. M. Bellare, P. Rogaway: "Random oracles are practical: a paradigm for designing efficient protocols", ACM Computer and Communication Security '93, ACM Press.
3. M. Bellare, A. Desai, D. Pointcheval, P. Rogaway: "Relations among notions of security for public-key encryption scheme", Crypto '98, Springer LNCS 1462.
4. R. Cramer, personal communication.
5. R. Cramer V. Shoup: "A practical cryptosystem, provably secure against chosen ciphertext attacks", Crypto '98, Springer LNCS 1462.
6. R. Cramer V. Shoup: "Design and analysis of practical public-key encryption schemes secure against adaptive chosen ciphertext attack", revised and extended version of [5], December 17, 2001, `http://eprint.iacr.org/2001/108/`.
7. R. Cramer V. Shoup: "Universal hash proofs and a paradigm for adaptive chosen ciphertext secure public-key encryption", preprint, 2nd version, December 12, 2001, `http://eprint.iacr.org/2001/085/`. (Extended abstract at Eurocrypt 2002.)
8. Y. Dodis, J. Katz, S. Xu, M. Yung: "Key-Insulated Public Key Cryptosystems", Eurocrypt 2002.
9. D. Dolev, C. Dwork, M. Naor: "Non-malleable cryptography", SIAM Journal of Computing, 2000. (Extended abstract at STOC '91, ACM Press.)
10. E. Kiltz, personal communication.

11. M. Naor, M, Yung: "Public-key cryptosystems provably secure against chosen ciphertext attacks", STOC '90, ACM Press.
12. C. Rackoff, D. Simon: "Non-interactive zero knowledge proof of knowledge and chosen ciphertext attacks", Crypto '91, Springer LNCS.
13. V. Shoup: "Using hash functions as a hedge against chosen ciphertext attack", Eurocrypt '00, Springer LNCS.

# Appendix

## A Properties of the Set $\mathrm{QR}_N$ – Proofs

In this section, we prove the Lemmas stated in Section 2. Consider the sets

$$\begin{aligned}\mathrm{QR}_P &= \{x \in \mathbf{Z}_P^* \,|\, \exists a \in \mathbf{Z}_P^* : a^2 \equiv x \,(\mathrm{mod}\, P)\},\\ \mathrm{QR}_Q &= \{x \in \mathbf{Z}_Q^* \,|\, \exists a \in \mathbf{Z}_Q^* : a^2 \equiv x \,(\mathrm{mod}\, Q)\}, \text{ and}\\ \mathrm{QR}_N &= \{x \in \mathbf{Z}_N^* \,|\, \exists a \in \mathbf{Z}_N^* : a^2 \equiv x \,(\mathrm{mod}\, N)\}\end{aligned}$$

of Quadratic Residues modulo $P$, $Q$ and $N$. Recall the following facts (which we don't prove in the current paper):

**Fact 1** *The sets $QR_N$, $QR_P$, and $QR_Q$ are multiplicative groups.*

**Fact 2** *$|QR_N| = pq$, $|QR_P| = p$, and $|QR_Q| = q$.*

**Fact 3** *Groups of prime order are cyclic.*

**Lemma 1.** *$QR_N$ has a nontrivial subgroup of order $p$ and a nontrivial subgroup of order $q$. Both subgroups are cyclic.*

*Proof.* Note that $x \in \mathbf{Z}_N$ is in $\mathrm{QR}_N$, if and only if $x$ is both a Quadratic Residue mod $P$ and a Quadratic Residue mod $Q$.

If $a \equiv 1 \,(\mathrm{mod}\, P)$ and $b \equiv 1 \,(\mathrm{mod}\, P)$, then $ab \equiv 1 \,(\mathrm{mod}\, P)$, and if both $a$ and $b$ are Quadratic Residues mod $Q$, then $ab$ is a Quadratic Residue mod $Q$ as well. Thus, the set

$$\{x \in \mathbf{Z}_N^* \mid \exists a \in \mathrm{QR}_Q : x \equiv a \bmod Q \text{ and } x \equiv 1 \bmod P\}$$

is a subgroup of $\mathrm{QR}_N$ of the order $|\mathrm{QR}_Q| = q$. Similarly, a subgroup of $\mathrm{QR}_N$ of the order $p$ exists. Groups of prime order are cyclic. □

**Lemma 2.** *$QR_N$ is cyclic. It consists of one element of the order 1, $(p-1)$ elements of the order $p$, $(q-1)$ elements of the order $q$, and $(p-1)(q-1)$ elements of the order $pq$.*

*Proof.* Consider $a, b \in \mathrm{QR}_N$ with $\mathrm{ord}(a) = p$, $\mathrm{ord}(b) = q$. Due to Lemma 1, such elements $a$ and $b$ exist; $\mathrm{ord}(ab) = \mathrm{lcm}(p, q) = pq$, thus $g = ab$ generates $\mathrm{QR}_N$.

Due to $\mathrm{ord}(g) = pq$ we have $\mathrm{ord}(g^0) = \mathrm{ord}(g^{ab}) = \mathrm{ord}(1) = 1$, $\mathrm{ord}(g^i) = pq \Leftrightarrow (i > 1$ and $\gcd(i, pq) = 1)$, $\mathrm{ord}(g^{kp}) = q$ for $k \in \{1, \ldots, q-1\}$, and $\mathrm{ord}(g^{lp}) = q$ for $l \in \{1, \ldots, p-1\}$. □

**Lemma 3.** *For every $x \in QR_N$: ord$(x) \in \{p, q\} \Rightarrow$ gcd$(x-1, N) \in \{P, Q\}$.*

*Proof.* From Lemma 2 and the proof of Lemma 1: $\mathrm{ord}(x) = q \Leftrightarrow X \equiv 1 \pmod{P} \Rightarrow \gcd(x-1, N) = P$. Similarly: $\mathrm{ord}(x) = p \Rightarrow \gcd(x-1, N) = Q$. □

Lemma 3 implies that an adversary who is able to find any $x \in \mathrm{QR}_N$ with $\mathrm{ord}(x) \notin \{1, pq\}$, can factorise $N$. Further, if $\mathrm{ord}(x) = pq$, then $\gcd(x-1, N) = 1$. An implication of Lemma 2 is that it is easy to find a random generator for $\mathrm{QR}_N$. Choose $x \in_R \mathbb{Z}_N^*$ and compute $g = x^2 \bmod N$. If $p$ and $q$ are large, $g$ is a generator for $\mathrm{QR}_N$ with overwhelming probability. In any case, $g$ is a generator if and only if $\mathrm{ord}(g) \notin \{1, p, q\}$; $\mathrm{ord}(g) = 1 \Leftrightarrow g = 1$, and Lemma 3 provides a way to check for $\mathrm{ord}(g) \notin \{p, q\}$.

**Lemma 4.** *Let $g$ be a generator for $QR_N$. For every $x \in \mathbb{Z}_{pq}$: ord$(g^x) \in \{p, q\} \Leftrightarrow$ gcd$(x, pq) \in \{p, q\}$.*

*Proof.* If $x \equiv p \pmod{pq}$, then $g^{qx} = 1$ and thus $\mathrm{ord}(g^x) = q$. If $\mathrm{ord}(g^x) = q$, then $(g^x)^p = 1 \Rightarrow xp \equiv 0 \bmod pq \Rightarrow x \equiv p \pmod{pq}$. Thus, $x \equiv p \pmod{pq} \Leftrightarrow \mathrm{ord}(g^x) = q$. Similarly, we get $x \equiv q \pmod{pq} \Leftrightarrow \mathrm{ord}(g^x) = p$. □

## B ACC-Security and Lunchtime-Security

Key decapsulation queries correspond to chosen ciphertext decryption queries in the public-key (PK) world. The key encapsulation query corresponds to the PK encryption query. Here, a plaintext is chosen by the adversary, the oracle either really encrypts that plaintext or it encrypts a random plaintext, and the adversary has to distinguish between real and random. Lunchtime (i.e. non-adaptive) security deals with *all* decryption queries *before* the encryption query. ACC attacks against PK cryptosystems deal with two phases of chosen ciphertext queries, the first *before* the encryption query, the second *after* the encryption query. (As mentioned in Footnote 1, some authors denote lunchtime security by "IND-CCA1" and ACC security by "IND-CCA2". Here "IND" means "indistinguishable". This notation has been introduced in [3].) In the second phase, one may not ask for the decryption of the result of the encryption query.

A definition for a lunchtime-secure KEM would require a minor modification of our definition for an ACC-secure KEM by asking the decapsulation queries *before* the encapsulation query. And a two-phase attack against a KEM with some decapsulation queries before and some after the encapsulation query – similar to the ACC attack against PK cryptosystems – can easily be simulated by our (one-phase) ACC attack.

## C The Proof for Factoring assumption ⇒ CDH assumption

*Proof (Theorem 1).* We describe an algorithm using a CDH oracle for $\mathrm{QR}_N$ as a tool to factorise $N$. For random inputs, the oracle succeeds with probability $\pi$.

- Choose $\beta \in_R \mathbb{Z}_{pq}$, $\alpha \in_R \mathbb{Z}_N^*$ and compute $g_2 = \alpha^2$ .
- Choose $u_1 \in_R \mathrm{QR}_N$ and compute $g = u_1^{2\beta}$.
- Use the CDH oracle to compute $u_2$ with $u_2^{2\beta} = g_2$.
- If $u_2^{\beta} \not\equiv \pm\alpha \pmod N$, print $\gcd(u_2^{\beta} - \alpha, N)$.

Since $\beta \in \mathbb{Z}_{pq}$ is a uniformly distributed random value (or statistically indistinguishable from uniform) so are the values $g, g_2, u_2 \in \mathrm{QR}_N$. With the probability $\pi$, we get a random square root $u_2^{\beta}$ of $g_2$. Two of the four square roots of $g_2$, namely $\pm\alpha$ are not useful, but if $\alpha \not\equiv \pm u_2^{\beta} \pmod N$, then $\gcd(u_2^{\beta} - \alpha, N) \in \{P, Q\}$ factorises $N$. □