OMAC: One-Key CBC MAC

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Abstract. In this paper, we present One-key CBC MAC (OMAC) and prove its security for arbitrary length messages. OMAC takes only one key, K (k bits) of a block cipher E. Previously, XCBC requires three keys, $(k + 2n)$ bits in total, and TMAC requires two keys, $(k + n)$ bits in total, where n denotes the block length of E.
The saving of the key length makes the security proof of OMAC substantially harder than those of XCBC and TMAC.

Key words: CBC MAC, block cipher, provable security

1 Introduction

1.1 Background

The CBC MAC [6, 7] is a well-known method to generate a message authentication code (MAC) based on a block cipher. Bellare, Kilian, and Rogaway proved the security of the CBC MAC for fixed message length mn bits, where n is the block length of the underlying block cipher E [1]. However, it is well known that the CBC MAC is *not* secure unless the message length is fixed.

Therefore, several variants of CBC MAC have been proposed for variable length messages.

First Encrypted MAC (EMAC) was proposed. It is obtained by encrypting the CBC MAC value by E again with a new key K_2. That is,

$$\mathrm{EMAC}_{K_1,K_2}(M) = E_{K_2}(\mathrm{CBC}_{K_1}(M)) \ ,$$

where M is a message, K_1 is the key of the CBC MAC and $\mathrm{CBC}_{K_1}(M)$ is the CBC MAC value of M [2]. Petrank and Rackoff then proved that EMAC is secure if the message length is a positive multiple of n [11] (Vaudenay showed another proof by using decorrelation theory [14]). Note that, however, EMAC requires two key schedulings of the underlying block cipher E.

Next Black and Rogaway proposed XCBC which requires only one key scheduling of the underlying block cipher E [3]. XCBC takes three keys: one block cipher key K_1, and two n-bit keys K_2 and K_3. XCBC is described as follows (see Fig. 1).

– If $|M| = mn$ for some $m > 0$, then XCBC computes exactly the same as the CBC MAC, except for XORing an n-bit key K_2 before encrypting the last block.

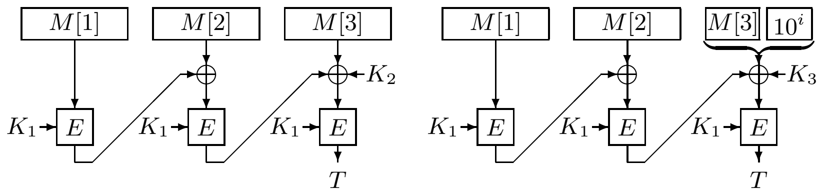

Fig. 1. Illustration of XCBC.

Table 1. Comparison of key length.

	XCBC [3]	TMAC [9]	OMAC (This paper)
key length	$(k+2n)$ bits	$(k+n)$ bits	k bits

– Otherwise, 10^i padding ($i = n-1-|M| \bmod n$) is appended to M and XCBC computes exactly the same as the CBC MAC for the padded message, except for XORing another n-bit key K_3 before encrypting the last block.

However, drawback of XCBC is that it requires three keys, $(k+2n)$ bits in total.

Finally Kurosawa and Iwata proposed Two-key CBC MAC (TMAC) [9]. TMAC takes two keys, $(k+n)$ bits in total: a block cipher key K_1 and an n-bit key K_2. TMAC is obtained from XCBC by replacing (K_2, K_3) with $(K_2 \cdot \mathtt{u}, K_2)$, where $\mathtt{u}$ is some non-zero constant and "$\cdot$" denotes multiplication in $\mathrm{GF}(2^n)$.

1.2 Our Contribution

In this paper, we present One-key CBC MAC (OMAC) and prove its security for arbitrary length messages. OMAC takes only one key, K of a block cipher E. The key length, k bits, is the minimum because the underlying block cipher must have a k-bit key K anyway. See Table 1 for a comparison with XCBC and TMAC (See Appendix A for a detailed comparison).

OMAC is a generic name for OMAC1 and OMAC2. OMAC1 is obtained from XCBC by replacing (K_2, K_3) with $(L \cdot \mathtt{u}, L \cdot \mathtt{u}^2)$ for some non-zero constant $\mathtt{u}$ in $\mathrm{GF}(2^n)$, where L is given by

$$L = E_K(0^n) \ .$$

OMAC2 is similarly obtained by using $(L \cdot \mathtt{u}, L \cdot \mathtt{u}^{-1})$. We can compute $L \cdot \mathtt{u}$, $L \cdot \mathtt{u}^{-1}$ and $L \cdot \mathtt{u}^2 = (L \cdot \mathtt{u}) \cdot \mathtt{u}$ efficiently by one shift and one conditional XOR from L, L and $L \cdot \mathtt{u}$, respectively.

OMAC1 (resp. OMAC2) is described as follows (see Fig. 2).

– If $|M| = mn$ for some $m > 0$, then OMAC computes exactly the same as the CBC MAC, except for XORing $L \cdot \mathtt{u}$ before encrypting the last block.

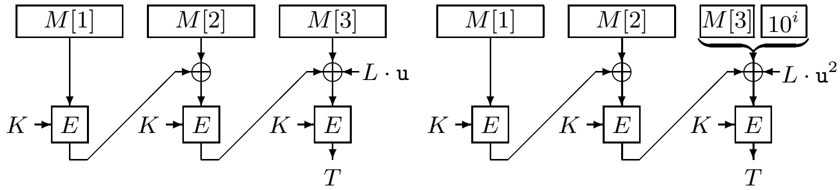

Fig. 2. Illustration of OMAC1. Note that $L = E_K(0^n)$. OMAC2 is obtained by replacing $L \cdot \mathtt{u}^2$ with $L \cdot \mathtt{u}^{-1}$ in the right figure.

– Otherwise, 10^i padding ($i = n-1-|M| \bmod n$) is appended to M and OMAC computes exactly the same as the CBC MAC for the padded message, except for XORing $L \cdot \mathtt{u}^2$ (resp. $L \cdot \mathtt{u}^{-1}$) before encrypting the last block.

Note that in TMAC, K_2 is a part of the key while in OMAC, L is not a part of the key and is generated from K.

This saving of the key length makes the security proof of OMAC substantially harder than that of TMAC, as shown below. In Fig. 2, suppose that $M[1] = 0^n$. Then the output of the first E_K is L. The same L always appears again at the last block. In general, such reuse of L would get one into trouble in the security proof.

(In OCB mode [13] and PMAC [5], $L = E_K(0^n)$ is also used as a key of a universal hash function. However, L appears as an output of some internal block cipher only with negligible probability.)

Nevertheless we prove that OMAC is as secure as XCBC, where the security analysis is in the concrete-security paradigm [1]. Further OMAC has all other nice properties which XCBC (and TMAC) has. That is, the domain of OMAC is $\{0,1\}^*$, it requires one key scheduling of the underlying block cipher E and $\max\{1, \lceil |M|/n \rceil\}$ block cipher invocations.

1.3 Other Related Work

Jaulmes, Joux and Valette proposed RMAC [8] which is an extension of EMAC. RMAC encrypts the CBC MAC value with $K_2 \oplus R$, where R is an n-bit random string and it is a part of the tag. That is,

$$\mathrm{RMAC}_{K_1,K_2}(M) = (E_{K_2 \oplus R}(\mathrm{CBC}_{K_1}(M)), R) \ .$$

They showed that the security of RMAC is beyond the birthday paradox limit. (XCBC, TMAC and OMAC are secure up to the birthday paradox limit.)

2 Preliminaries

2.1 Notation

We use similar notation as in [13, 5]. For a set A, $x \stackrel{R}{\leftarrow} A$ means that x is chosen from A uniformly at random. If $a, b \in \{0,1\}^*$ are equal-length strings then $a \oplus b$ is their bitwise XOR. If $a, b \in \{0,1\}^*$ are strings then $a \circ b$ denote their concatenation. For simplicity, we sometimes write ab for $a \circ b$ if there is no confusion.

For an n-bit string $a = a_{n-1} \cdots a_1 a_0 \in \{0,1\}^n$, let $a \ll 1 = a_{n-2} \cdots a_1 a_0 0$ denote the n-bit string which is a left shift of a by 1 bit, while $a \gg 1 = 0 a_{n-1} \cdots a_2 a_1$ denote the n-bit string which is a right shift of a by 1 bit.

If $a \in \{0,1\}^*$ is a string then $|a|$ denotes its length in bits. For any bit string $a \in \{0,1\}^*$ such that $|a| \leq n$, we let

$$\mathtt{pad}_n(a) = \begin{cases} a10^{n-|a|-1} & \text{if } |a| < n, \\ a & \text{if } |a| = n. \end{cases} \tag{1}$$

Define $\|a\|_n = \max\{1, \lceil |a|/n \rceil\}$, where the empty string counts as one block. In pseudocode, we write "Partition M into $M[1] \cdots M[m]$" as shorthand for "Let $m = \|M\|_n$, and let $M[1], \ldots, M[m]$ be bit strings such that $M[1] \cdots M[m] = M$ and $|M[i]| = n$ for $1 \leq i < m$."

2.2 CBC MAC

The block cipher E is a function $E : \mathcal{K}_E \times \{0,1\}^n \to \{0,1\}^n$, where each $E(K, \cdot) = E_K(\cdot)$ is a permutation on $\{0,1\}^n$, $\mathcal{K}_E$ is the set of possible keys and n is the block length.

The CBC MAC [6, 7] is the simplest and most well-known algorithm to make a MAC from a block cipher E. Let $M = M[1] \circ M[2] \circ \cdots \circ M[m]$ be a message string, where $|M[1]| = |M[2]| = \cdots = |M[m]| = n$. Then $\mathrm{CBC}_K(M)$, the CBC MAC of M under key K, is defined as $Y[m]$, where

$$Y[i] = E_K(M[i] \oplus Y[i-1])$$

for $i = 1, \ldots, m$ and $Y[0] = 0^n$. Bellare, Kilian and Rogaway proved the security of the CBC MAC for fixed message length mn bits [1].

2.3 The Field with 2^n Points

We interchangeably think of a point a in $\mathrm{GF}(2^n)$ in any of the following ways: (1) as an abstract point in a field; (2) as an n-bit string $a_{n-1} \cdots a_1 a_0 \in \{0,1\}^n$; (3) as a formal polynomial $a(\mathtt{u}) = a_{n-1}\mathtt{u}^{n-1} + \cdots + a_1\mathtt{u} + a_0$ with binary coefficients.

To add two points in $\mathrm{GF}(2^n)$, take their bitwise XOR. We denote this operation by $a \oplus b$.

To multiply two points, fix some irreducible polynomial $f(\mathtt{u})$ having binary coefficients and degree n. To be concrete, choose the lexicographically first polynomial among the irreducible degree n polynomials having a minimum number of coefficients. We list some indicated polynomials (See [10, Chapter 10] for other polynomials).

$$\begin{cases} f(\mathtt{u}) = \mathtt{u}^{64} + \mathtt{u}^4 + \mathtt{u}^3 + \mathtt{u} + 1 & \text{for } n = 64, \\ f(\mathtt{u}) = \mathtt{u}^{128} + \mathtt{u}^7 + \mathtt{u}^2 + \mathtt{u} + 1 & \text{for } n = 128, \text{and} \\ f(\mathtt{u}) = \mathtt{u}^{256} + \mathtt{u}^{10} + \mathtt{u}^5 + \mathtt{u}^2 + 1 & \text{for } n = 256. \end{cases}$$

To multiply two points $a \in \mathrm{GF}(2^n)$ and $b \in \mathrm{GF}(2^n)$, regard a and b as polynomials $a(\mathtt{u}) = a_{n-1}\mathtt{u}^{n-1} + \cdots + a_1\mathtt{u} + a_0$ and $b(\mathtt{u}) = b_{n-1}\mathtt{u}^{n-1} + \cdots + b_1\mathtt{u} + b_0$, form their product $c(\mathtt{u})$ where one adds and multiplies coefficients in GF(2), and take the remainder when dividing $c(\mathtt{u})$ by $f(\mathtt{u})$.

Note that it is particularly easy to multiply a point $a \in \{0,1\}^n$ by $\mathtt{u}$. For example, if $n = 128$,

$$a \cdot \mathtt{u} = \begin{cases} a \ll 1 & \text{if } a_{127} = 0, \\ (a \ll 1) \oplus 0^{120}10000111 & \text{otherwise.} \end{cases} \tag{2}$$

Also, note that it is easy to divide a point $a \in \{0,1\}^n$ by $\mathtt{u}$, meaning that one multiplies a by the multiplicative inverse of $\mathtt{u}$ in the field: $a \cdot \mathtt{u}^{-1}$. For example, if $n = 128$,

$$a \cdot \mathtt{u}^{-1} = \begin{cases} a \gg 1 & \text{if } a_0 = 0, \\ (a \gg 1) \oplus 10^{120}1000011 & \text{otherwise.} \end{cases} \tag{3}$$

3 Basic Construction

In this section, we show a basic construction of OMAC-family.

OMAC-family is defined by a block cipher $E : \mathcal{K}_E \times \{0,1\}^n \to \{0,1\}^n$, an n-bit constant $\mathtt{Cst}$, a universal hash function $H : \{0,1\}^n \times X \to \{0,1\}^n$, and two distinct constants $\mathtt{Cst}_1, \mathtt{Cst}_2 \in X$, where X is the finite domain of H.

H, $\mathtt{Cst}_1$ and $\mathtt{Cst}_2$ must satisfy the following conditions while $\mathtt{Cst}$ is arbitrary. We write $H_L(\cdot)$ for $H(L, \cdot)$.

1. For any $y \in \{0,1\}^n$, the number of $L \in \{0,1\}^n$ such that $H_L(\mathtt{Cst}_1) = y$ is at most $\epsilon_1 \cdot 2^n$ for some sufficiently small ϵ_1.
2. For any $y \in \{0,1\}^n$, the number of $L \in \{0,1\}^n$ such that $H_L(\mathtt{Cst}_2) = y$ is at most $\epsilon_2 \cdot 2^n$ for some sufficiently small ϵ_2.
3. For any $y \in \{0,1\}^n$, the number of $L \in \{0,1\}^n$ such that $H_L(\mathtt{Cst}_1) \oplus H_L(\mathtt{Cst}_2) = y$ is at most $\epsilon_3 \cdot 2^n$ for some sufficiently small ϵ_3.
4. For any $y \in \{0,1\}^n$, the number of $L \in \{0,1\}^n$ such that $H_L(\mathtt{Cst}_1) \oplus L = y$ is at most $\epsilon_4 \cdot 2^n$ for some sufficiently small ϵ_4.
5. For any $y \in \{0,1\}^n$, the number of $L \in \{0,1\}^n$ such that $H_L(\mathtt{Cst}_2) \oplus L = y$ is at most $\epsilon_5 \cdot 2^n$ for some sufficiently small ϵ_5.
6. For any $y \in \{0,1\}^n$, the number of $L \in \{0,1\}^n$ such that $H_L(\mathtt{Cst}_1) \oplus H_L(\mathtt{Cst}_2) \oplus L = y$ is at most $\epsilon_6 \cdot 2^n$ for some sufficiently small ϵ_6.

Remark 3.1. Property 1 and 2 says that $H_L(\mathtt{Cst}_1)$ and $H_L(\mathtt{Cst}_2)$ are almost uniformly distributed. Property 3 is satisfied by AXU (almost XOR universal) hash functions [12]. Property 4, 5, 6 are new requirements introduced here.

The algorithm of OMAC-family is described in Fig. 3 and illustrated in Fig. 4, where $\mathtt{pad}_n(\cdot)$ is defined in (1).

The key space $\mathcal{K}$ of OMAC-family is $\mathcal{K} = \mathcal{K}_E$. It takes a key $K \in \mathcal{K}_E$ and a message $M \in \{0,1\}^*$, and returns a string in $\{0,1\}^n$.

Algorithm OMAC-family$_K(M)$
$L \leftarrow E_K(\texttt{Cst})$
$Y[0] \leftarrow 0^n$
Partition M into $M[1] \cdots M[m]$
for $i \leftarrow 1$ **to** $m-1$ **do**
    $X[i] \leftarrow M[i] \oplus Y[i-1]$
    $Y[i] \leftarrow E_K(X[i])$
$X[m] \leftarrow \texttt{pad}_n(M[m]) \oplus Y[m-1]$
if $|M[m]| = n$ **then** $X[m] \leftarrow X[m] \oplus H_L(\texttt{Cst}_1)$
    else $X[m] \leftarrow X[m] \oplus H_L(\texttt{Cst}_2)$
$T \leftarrow E_K(X[m])$
return T

Fig. 3. Definition of OMAC-family.

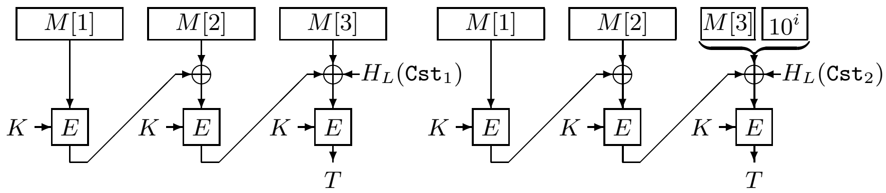

Fig. 4. Illustration of OMAC-family.

4 Proposed Specification

In this section, we present two specifications of OMAC-family: OMAC1 and OMAC2. We use OMAC as a generic name for OMAC1 and OMAC2.

In OMAC1 we let $\texttt{Cst} = 0^n$, $H_L(x) = L \cdot x$, $\texttt{Cst}_1 = \texttt{u}$ and $\texttt{Cst}_2 = \texttt{u}^2$, where "$\cdot$" denotes multiplication over GF(2^n). Equivalently, $L = E_K(0^n)$, $H_L(\texttt{Cst}_1) = L \cdot \texttt{u}$ and $H_L(\texttt{Cst}_2) = L \cdot \texttt{u}^2$. OMAC2 is the same as OMAC1 except for $\texttt{Cst}_2 = \texttt{u}^{-1}$ instead of $\texttt{Cst}_2 = \texttt{u}^2$. Equivalently, $L = E_K(0^n)$, $H_L(\texttt{Cst}_1) = L \cdot \texttt{u}$ and $H_L(\texttt{Cst}_2) = L \cdot \texttt{u}^{-1}$.

Note that $L \cdot \texttt{u}$, $L \cdot \texttt{u}^{-1}$ and $L \cdot \texttt{u}^2 = (L \cdot \texttt{u}) \cdot \texttt{u}$ can be computed efficiently by one shift and one conditional XOR from L, L and $L \cdot \texttt{u}$, respectively as shown in (2) and (3). It is easy to see that the conditions in Sec. 3 are satisfied for $\epsilon_1 = \cdots = \epsilon_6 = 2^{-n}$ in OMAC1 and OMAC2.

OMAC1 and OMAC2 are described in Fig. 5 and illustrated in Fig. 2.

5 Security of OMAC-Family

5.1 Security Definitions

Let Perm(n) denote the set of all permutations on $\{0,1\}^n$. We say that P is a random permutation if P is randomly chosen from Perm(n).

The security of a block cipher E can be quantified as $\texttt{Adv}_E^{\texttt{prp}}(t, q)$, the maximum advantage that an adversary $\mathcal{A}$ can obtain when trying to distinguish

Algorithm $\mathrm{OMAC1}_K(M)$	**Algorithm** $\mathrm{OMAC2}_K(M)$
$L \leftarrow E_K(0^n)$	$L \leftarrow E_K(0^n)$
$Y[0] \leftarrow 0^n$	$Y[0] \leftarrow 0^n$
Partition M into $M[1]\cdots M[m]$	Partition M into $M[1]\cdots M[m]$
for $i \leftarrow 1$ **to** $m-1$ **do**	**for** $i \leftarrow 1$ **to** $m-1$ **do**
$\quad X[i] \leftarrow M[i] \oplus Y[i-1]$	$\quad X[i] \leftarrow M[i] \oplus Y[i-1]$
$\quad Y[i] \leftarrow E_K(X[i])$	$\quad Y[i] \leftarrow E_K(X[i])$
$X[m] \leftarrow \mathtt{pad}_n(M[m]) \oplus Y[m-1]$	$X[m] \leftarrow \mathtt{pad}_n(M[m]) \oplus Y[m-1]$
if $\lvert M[m]\rvert = n$	**if** $\lvert M[m]\rvert = n$
$\quad$ **then** $X[m] \leftarrow X[m] \oplus L \cdot \mathtt{u}$	$\quad$ **then** $X[m] \leftarrow X[m] \oplus L \cdot \mathtt{u}$
$\quad$ **else** $X[m] \leftarrow X[m] \oplus L \cdot \mathtt{u}^2$	$\quad$ **else** $X[m] \leftarrow X[m] \oplus L \cdot \mathtt{u}^{-1}$
$T \leftarrow E_K(X[m])$	$T \leftarrow E_K(X[m])$
return T	**return** T

Fig. 5. Description of OMAC1 and OMAC2.

$E_K(\cdot)$ (with a randomly chosen key K) from a random permutation $P(\cdot)$, when allowed computation time t and q queries to an oracle (which is either $E_K(\cdot)$ or $P(\cdot)$). This advantage is defined as follows.

$$\begin{cases} \mathtt{Adv}_E^{\mathsf{prp}}(\mathcal{A}) \stackrel{\text{def}}{=} \left|\Pr(K \stackrel{R}{\leftarrow} \mathcal{K}_E : \mathcal{A}^{E_K(\cdot)} = 1) - \Pr(P \stackrel{R}{\leftarrow} \mathrm{Perm}(n) : \mathcal{A}^{P(\cdot)} = 1)\right| \\ \mathtt{Adv}_E^{\mathsf{prp}}(t,q) \stackrel{\text{def}}{=} \max_{\mathcal{A}} \{\mathtt{Adv}_E^{\mathsf{prp}}(\mathcal{A})\} \end{cases}$$

We say that a block cipher E is secure if $\mathtt{Adv}_E^{\mathsf{prp}}(t,q)$ is sufficiently small.

Similarly, a MAC algorithm is a map $F : \mathcal{K}_F \times \{0,1\}^* \to \{0,1\}^n$, where $\mathcal{K}_F$ is a set of keys and we write $F_K(\cdot)$ for $F(K,\cdot)$. We say that an adversary $\mathcal{A}^{F_K(\cdot)}$ *forges* if $\mathcal{A}$ outputs $(M, F_K(M))$ where $\mathcal{A}$ never queried M to its oracle $F_K(\cdot)$. Then we define the advantage as

$$\begin{cases} \mathtt{Adv}_F^{\mathsf{mac}}(\mathcal{A}) \stackrel{\text{def}}{=} \Pr(K \stackrel{R}{\leftarrow} \mathcal{K}_F : \mathcal{A}^{F_K(\cdot)} \text{ forges}) \\ \mathtt{Adv}_F^{\mathsf{mac}}(t,q,\mu) \stackrel{\text{def}}{=} \max_{\mathcal{A}} \{\mathtt{Adv}_F^{\mathsf{mac}}(\mathcal{A})\} \end{cases}$$

where the maximum is over all adversaries who run in time at most t, make at most q queries, and each query is at most μ bits. We say that a MAC algorithm is secure if $\mathtt{Adv}_F^{\mathsf{mac}}(t,q,\mu)$ is sufficiently small.

Let $\mathrm{Rand}(*,n)$ denote the set of all functions from $\{0,1\}^*$ to $\{0,1\}^n$. This set is given a probability measure by asserting that a random element R of $\mathrm{Rand}(*,n)$ associates to each string $M \in \{0,1\}^*$ a random string $R(M) \in \{0,1\}^n$. Then we define the advantage as

$$\begin{cases} \mathtt{Adv}_F^{\mathsf{viprf}}(\mathcal{A}) \stackrel{\text{def}}{=} \left|\Pr(K \stackrel{R}{\leftarrow} \mathcal{K}_F : \mathcal{A}^{F_K(\cdot)} = 1) - \Pr(R \stackrel{R}{\leftarrow} \mathrm{Rand}(*,n) : \mathcal{A}^{R(\cdot)} = 1)\right| \\ \mathtt{Adv}_F^{\mathsf{viprf}}(t,q,\mu) \stackrel{\text{def}}{=} \max_{\mathcal{A}} \left\{\mathtt{Adv}_F^{\mathsf{viprf}}(\mathcal{A})\right\} \end{cases}$$

where the maximum is over all adversaries who run in time at most t, make at most q queries, and each query is at most μ bits. We say that a MAC algorithm is pseudorandom if $\mathtt{Adv}_F^{\mathsf{viprf}}(t,q,\mu)$ is sufficiently small (viprf stands for Variable-length Input PseudoRandom Function).

Without loss of generality, adversaries are assumed to never ask a query outside the domain of the oracle, and to never repeat a query.

5.2 Theorem Statements

We first prove that OMAC-family is pseudorandom if the underlying block cipher is a random permutation P (information-theoretic result). This proof is much harder than the previous works because of the reuse of L as explained Sec. 1.2.

Lemma 5.1 (Main Lemma for OMAC-family). *Suppose that H, $\mathtt{Cst}_1$ and $\mathtt{Cst}_2$ satisfy the conditions in Sec. 3 for some sufficiently small $\epsilon_1, \ldots, \epsilon_6$, and let $\mathtt{Cst}$ be an arbitrarily n-bit constant. Suppose that a random permutation $P \in Perm(n)$ is used in OMAC-family as the underlying block cipher. Let $\mathcal{A}$ be an adversary which asks at most q queries, and each query is at most nm bits (m is the maximum number of blocks in each query). Assume $m \le 2^n/4$. Then*

$$\Big|\Pr(P \stackrel{R}{\leftarrow} Perm(n) : \mathcal{A}^{\text{OMAC-family}_P(\cdot)} = 1) \\ - \Pr(R \stackrel{R}{\leftarrow} Rand(*,n) : \mathcal{A}^{R(\cdot)} = 1)\Big| \le \frac{q^2}{2} \cdot \left(\frac{7m^2+2}{2^n} + 3m^2\epsilon\right) \ , \tag{4}$$

where $\epsilon = \max\{\epsilon_1, \ldots, \epsilon_6\}$.

A proof is given in the next section.

The following results hold for both OMAC1 and OMAC2. First, we obtain the following lemma by substituting $\epsilon = 2^{-n}$ in Lemma 5.1.

Lemma 5.2 (Main Lemma for OMAC). *Suppose that a random permutation $P \in Perm(n)$ is used in OMAC as the underlying block cipher. Let $\mathcal{A}$ be an adversary which asks at most q queries, and each query is at most nm bits. Assume $m \le 2^n/4$. Then*

$$\Big|\Pr(P \stackrel{R}{\leftarrow} Perm(n) : \mathcal{A}^{\mathrm{OMAC}_P(\cdot)} = 1) \\ - \Pr(R \stackrel{R}{\leftarrow} Rand(*,n) : \mathcal{A}^{R(\cdot)} = 1)\Big| \le \frac{(5m^2+1)q^2}{2^n} \ .$$

We next show that OMAC is pseudorandom if the underlying block cipher E is secure. It is standard to pass to this complexity-theoretic result from Lemma 5.2. (For example, see [1, Section 3.2] for the proof technique. In [1, Section 3.2], it is shown that a complexity-theoretic advantage of the CBC MAC is obtained from its information-theoretic advantage.)

Corollary 5.1. *Let $E : \mathcal{K}_E \times \{0,1\}^n \to \{0,1\}^n$ be the underlying block cipher used in OMAC. Then*

$$\mathtt{Adv}_{\mathrm{OMAC}}^{\mathsf{viprf}}(t,q,nm) \le \frac{(5m^2+1)q^2}{2^n} + \mathtt{Adv}_E^{\mathsf{prp}}(t',q') \ ,$$

where $t' = t + O(mq)$ and $q' = mq + 1$.

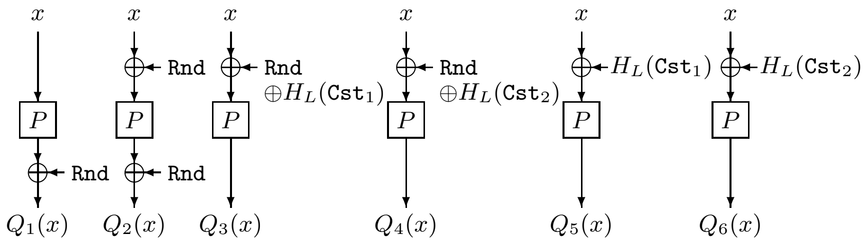

Fig. 6. Illustrations of Q_1, Q_2 Q_3, Q_4, Q_5 and Q_6. Note that $L = P(\mathtt{Cst})$.

Finally we show that OMAC is secure as a MAC algorithm from Corollary 5.1 in the usual way. (For example, see [1, Proposition 2.7] for the proof technique. In [1, Proposition 2.7], it is shown that pseudorandom functions are secure MACs.)

Theorem 5.1. *Let $E : \mathcal{K}_E \times \{0,1\}^n \to \{0,1\}^n$ be the underlying block cipher used in OMAC. Then*

$$\mathtt{Adv}^{\mathsf{mac}}_{\mathrm{OMAC}}(t, q, nm) \le \frac{(5m^2+1)q^2+1}{2^n} + \mathtt{Adv}^{\mathsf{prp}}_{E}(t', q') \ ,$$

where $t' = t + O(mq)$ and $q' = mq + 1$.

5.3 Proof of Main Lemma for OMAC-family

Let H, $\mathtt{Cst}_1$ and $\mathtt{Cst}_2$ satisfy the conditions in Sec. 3 for some sufficiently small $\epsilon_1, \ldots, \epsilon_6$, and $\mathtt{Cst}$ be an arbitrarily n-bit constant. For a random permutation $P \in \mathrm{Perm}(n)$ and a random n-bit string $\mathtt{Rnd} \in \{0,1\}^n$, define

$$\begin{cases} Q_1(x) \stackrel{\mathrm{def}}{=} P(x) \oplus \mathtt{Rnd}, & Q_2(x) \stackrel{\mathrm{def}}{=} P(x \oplus \mathtt{Rnd}) \oplus \mathtt{Rnd}, \\ Q_3(x) \stackrel{\mathrm{def}}{=} P(x \oplus \mathtt{Rnd} \oplus H_L(\mathtt{Cst}_1)), & Q_4(x) \stackrel{\mathrm{def}}{=} P(x \oplus \mathtt{Rnd} \oplus H_L(\mathtt{Cst}_2)), \\ Q_5(x) \stackrel{\mathrm{def}}{=} P(x \oplus H_L(\mathtt{Cst}_1)) \text{ and} & Q_6(x) \stackrel{\mathrm{def}}{=} P(x \oplus H_L(\mathtt{Cst}_2)), \end{cases} \quad (5)$$

where $L = P(\mathtt{Cst})$. See Fig. 6 for illustrations.

We first show that $Q_1(\cdot)$, $Q_2(\cdot)$, $Q_3(\cdot)$, $Q_4(\cdot)$, $Q_5(\cdot)$, $Q_6(\cdot)$ are indistinguishable from a pair of six independent random permutations $P_1(\cdot)$, $P_2(\cdot)$, $P_3(\cdot)$, $P_4(\cdot)$, $P_5(\cdot)$, $P_6(\cdot)$.

Lemma 5.3. *Let $\mathcal{A}$ be an adversary which asks at most q queries in total. Then*

$$\Big| \Pr(P \stackrel{R}{\leftarrow} Perm(n); \mathtt{Rnd} \stackrel{R}{\leftarrow} \{0,1\}^n : \mathcal{A}^{Q_1(\cdot),\ldots,Q_6(\cdot)} = 1) \\ - \Pr(P_1, \ldots, P_6 \stackrel{R}{\leftarrow} Perm(n) : \mathcal{A}^{P_1(\cdot),\ldots,P_6(\cdot)} = 1) \Big| \le \frac{3q^2}{2} \cdot \left(\frac{1}{2^n} + \epsilon \right) \ ,$$

where $\epsilon = \max\{\epsilon_1, \ldots, \epsilon_6\}$.

Algorithm $\mathrm{MOMAC}_{P_1,P_2,P_3,P_4,P_5,P_6}(M)$
Partition M into $M[1] \cdots M[m]$
if $m \geq 2$ **then**
  $X[1] \leftarrow M[1]$
  $Y[1] \leftarrow P_1(X[1])$
  for $i \leftarrow 2$ **to** $m-1$ **do**
    $X[i] \leftarrow M[i] \oplus Y[i-1]$
    $Y[i] \leftarrow P_2(X[i])$
  $X[m] \leftarrow \mathtt{pad}_n(M[m]) \oplus Y[m-1]$
  if $|M[m]| = n$ **then** $T \leftarrow P_3(X[m])$
        else $T \leftarrow P_4(X[m])$
if $m = 1$ **then**
  $X[m] \leftarrow \mathtt{pad}_n(M[m])$
  if $|M[m]| = n$ **then** $T \leftarrow P_5(X[m])$
        else $T \leftarrow P_6(X[m])$
return T

Fig. 7. Definition of MOMAC.

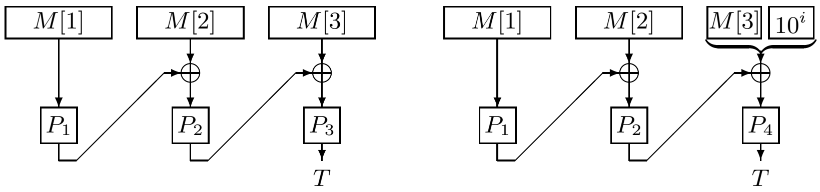

Fig. 8. Illustration of MOMAC for $|M| > n$.

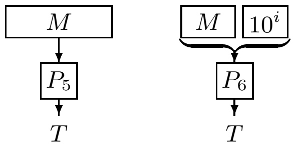

Fig. 9. Illustration of MOMAC for $|M| \leq n$.

A proof is given in Appendix B.

Next we define MOMAC (Modified OMAC). It uses six independent random permutations $P_1, P_2, P_3, P_4, P_5, P_6 \in \mathrm{Perm}(n)$. The algorithm $\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)$ is described in Fig. 7 and illustrated in Fig. 8 and Fig. 9.

We prove that MOMAC is pseudorandom.

Lemma 5.4. *Let $\mathcal{A}$ be an adversary which asks at most q queries, and each query is at most nm bits. Assume $m \leq 2^n/4$. Then*

$$\Big| \Pr(P_1, \ldots, P_6 \stackrel{R}{\leftarrow} \mathit{Perm}(n) : \mathcal{A}^{\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)} = 1) - \Pr(R \stackrel{R}{\leftarrow} \mathit{Rand}(*, n) : \mathcal{A}^{R(\cdot)} = 1) \Big| \leq \frac{(2m^2+1)q^2}{2^n} \ .$$

A proof is given in Appendix C.

Algorithm $\mathcal{B}_{\mathcal{A}}^{\mathcal{O}_1,\ldots,\mathcal{O}_6}$
1: **When** $\mathcal{A}$ **asks its** r**-th query** $M^{(r)}$**:**
2: $\quad T^{(r)} \leftarrow \mathrm{MOMAC}_{\mathcal{O}_1,\ldots,\mathcal{O}_6}(M^{(r)})$
3: $\quad$ **return** $T^{(r)}$
4: **When** $\mathcal{A}$ **halts and outputs** b**:**
5: $\quad$ **output** b

Fig. 10. Algorithm $\mathcal{B}_{\mathcal{A}}$. Note that for $1 \le i \le 6$, $\mathcal{O}_i$ is either P_i or Q_i

The next lemma shows that OMAC-family$_P(\cdot)$ and $\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)$ are indistinguishable.

Lemma 5.5. *Let $\mathcal{A}$ be an adversary which asks at most q queries, and each query is at most nm bits. Assume $m \le 2^n/4$. Then*

$$\Big|\Pr(P \stackrel{R}{\leftarrow} Perm(n) : \mathcal{A}^{\text{OMAC-family}_P(\cdot)} = 1) \\ - \Pr(P_1,\ldots,P_6 \stackrel{R}{\leftarrow} Perm(n) : \mathcal{A}^{\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)} = 1)\Big| \le \frac{3m^2q^2}{2} \cdot \left(\frac{1}{2^n} + \epsilon\right) .$$

Proof. Suppose that there exists an adversary $\mathcal{A}$ such that

$$\Big|\Pr(P \stackrel{R}{\leftarrow} \mathrm{Perm}(n) : \mathcal{A}^{\text{OMAC-family}_P(\cdot)} = 1) \\ - \Pr(P_1,\ldots,P_6 \stackrel{R}{\leftarrow} \mathrm{Perm}(n) : \mathcal{A}^{\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)} = 1)\Big| > \frac{3m^2q^2}{2} \cdot \left(\frac{1}{2^n} + \epsilon\right) .$$

By using $\mathcal{A}$, we show a construction of an adversary $\mathcal{B}_{\mathcal{A}}$ such that:

- $\mathcal{B}_{\mathcal{A}}$ asks at most mq queries, and
- $\Big|\Pr(P \stackrel{R}{\leftarrow} \mathrm{Perm}(n) : \mathcal{B}_{\mathcal{A}}^{Q_1(\cdot),\ldots,Q_6(\cdot)} = 1) \\ - \Pr(P_1,\ldots,P_6 \stackrel{R}{\leftarrow} \mathrm{Perm}(n) : \mathcal{B}_{\mathcal{A}}^{P_1(\cdot),\ldots,P_6(\cdot)} = 1)\Big| > \frac{3m^2q^2}{2} \cdot \left(\frac{1}{2^n} + \epsilon\right),$

which contradicts Lemma 5.3.

Let $\mathcal{O}_1(\cdot),\ldots,\mathcal{O}_6(\cdot)$ be $\mathcal{B}_{\mathcal{A}}$'s oracles. The construction of $\mathcal{B}_{\mathcal{A}}$ is given in Fig. 10.

When $\mathcal{A}$ asks $M^{(r)}$, then $\mathcal{B}_{\mathcal{A}}$ computes $T^{(r)} = \mathrm{MOMAC}_{\mathcal{O}_1,\ldots,\mathcal{O}_6}(M^{(r)})$ as if the underlying random permutations are $\mathcal{O}_1,\ldots,\mathcal{O}_6$, and returns $T^{(r)}$. When $\mathcal{A}$ halts and outputs b, then $\mathcal{B}_{\mathcal{A}}$ outputs b.

Now we see that:

- $\mathcal{B}_{\mathcal{A}}$ asks at most mq queries to its oracles, since $\mathcal{A}$ asks at most q queries, and each query is at most nm bits.
- $\Pr(P_1,\ldots,P_6 \stackrel{R}{\leftarrow} \mathrm{Perm}(n) : \mathcal{B}_{\mathcal{A}}^{P_1(\cdot),\ldots,P_6(\cdot)} = 1) \\ = \Pr(P_1,\ldots,P_6 \stackrel{R}{\leftarrow} \mathrm{Perm}(n) : \mathcal{A}^{\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)} = 1),$
  since $\mathcal{B}_{\mathcal{A}}$ gives $\mathcal{A}$ a perfect simulation of $\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)$ if $\mathcal{O}_i(\cdot) = P_i(\cdot)$ for $1 \le i \le 6$.

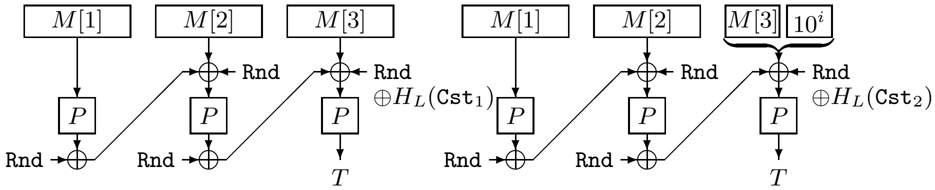

Fig. 11. Computation of $\mathcal{B}_\mathcal{A}$ when $\mathcal{O}_i = Q_i$ for $1 \le i \le 6$, and $|M| > n$.

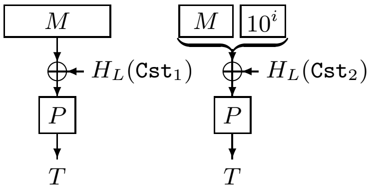

Fig. 12. Computation of $\mathcal{B}_\mathcal{A}$ when $\mathcal{O}_i = Q_i$ for $1 \le i \le 6$, and $|M| \le n$.

– $\Pr(P \xleftarrow{R} \mathrm{Perm}(n) : \mathcal{B}_\mathcal{A}^{Q_1(\cdot),\ldots,Q_6(\cdot)} = 1)$
$= \Pr(P \xleftarrow{R} \mathrm{Perm}(n) : \mathcal{A}^{\mathrm{OMAC}_P(\cdot)} = 1)$,
since $\mathcal{B}_\mathcal{A}$ gives $\mathcal{A}$ a perfect simulation of $\mathrm{OMAC}_P(\cdot)$ if $\mathcal{O}_i(\cdot) = Q_i(\cdot)$ for $1 \le i \le 6$. See Fig. 11 and Fig. 12. Note that `Rnd` is canceled in Fig. 11.

This concludes the proof of the lemma. □

We finally give a proof of Main Lemma for OMAC-family.

Proof (of Lemma 5.1). By the triangle inequality, the left hand side of (4) is at most

$$\Big|\Pr(P_1,\ldots,P_6 \xleftarrow{R} \mathrm{Perm}(n) : \mathcal{A}^{\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)} = 1) - \Pr(R \xleftarrow{R} \mathrm{Rand}(*,n) : \mathcal{A}^{R(\cdot)} = 1)\Big| \qquad (6)$$

$$+ \Big|\Pr(P \xleftarrow{R} \mathrm{Perm}(n) : \mathcal{A}^{\text{OMAC-family}_P(\cdot)} = 1) - \Pr(P_1,\ldots,P_6 \xleftarrow{R} \mathrm{Perm}(n) : \mathcal{A}^{\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)} = 1)\Big| \ . \qquad (7)$$

Lemma 5.4 gives us an upper bound on (6) and Lemma 5.5 gives us an upper bound on (7). Therefore the bound follows since

$$\frac{(2m^2+1)q^2}{2^n} + \frac{3m^2q^2}{2} \cdot \left(\frac{1}{2^n} + \epsilon\right) = \frac{q^2}{2} \cdot \left(\frac{7m^2+2}{2^n} + 3m^2\epsilon\right) \ .$$

This concludes the proof of the lemma. □

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A Discussions

A.1 Design Rationale

Our choice for OMAC1 is $\mathtt{Cst} = 0^n$, $H_L(x) = L \cdot x$, $\mathtt{Cst}_1 = \mathtt{u}$ and $\mathtt{Cst}_2 = \mathtt{u}^2$, where "$\cdot$" denotes multiplication over $\mathrm{GF}(2^n)$. Similarly, our choice for OMAC2 is $\mathtt{Cst} = 0^n$, $H_L(x) = L \cdot x$, $\mathtt{Cst}_1 = \mathtt{u}$ and $\mathtt{Cst}_2 = \mathtt{u}^{-1}$. Below, we list reasons of this choice.

- One might try to use $\mathtt{Cst}_1 = 1$ instead of $\mathtt{Cst}_1 = \mathtt{u}$. In this case, the fourth condition in Sec. 3 is not satisfied, and in fact, the scheme can be easily attacked. Similarly, if one uses $\mathtt{Cst}_2 = 1$ instead of $\mathtt{Cst}_2 = \mathtt{u}^2$ or $\mathtt{Cst}_2 = \mathtt{u}^{-1}$, the fifth condition in Sec. 3 is not satisfied, and the scheme can be easily attacked. Therefore, we can not use "1" as a constant.
- For OMAC1, we adopted $\mathtt{u}$ and $\mathtt{u}^2$ as $\mathtt{Cst}_1$ and $\mathtt{Cst}_2$, since $L \cdot \mathtt{u}$ and $L \cdot \mathtt{u}^2 = (L \cdot \mathtt{u}) \cdot \mathtt{u}$ can be computed efficiently by one left shift and one conditional XOR from L and $L \cdot \mathtt{u}$, respectively, as shown in (2). Note that this choice requires only a left shift. This would ease the implementation of OMAC1, especially in hardware.
- For OMAC2, we adopted $\mathtt{u}^{-1}$ instead of $\mathtt{u}^2$ as $\mathtt{Cst}_2$. It requires one right shift to compute $L \cdot \mathtt{u}^{-1}$ instead of one left shift to compute $(L \cdot \mathtt{u}) \cdot \mathtt{u}$. This would allow to compute both $L \cdot \mathtt{u}$ and $L \cdot \mathtt{u}^{-1}$ from L simultaneously if both left shift and right shift are available (for example, the underlying block cipher uses both shifts).

A.2 On Standard Key Separation Technique

For XCBC, assume that we want to use a single key K of E, where E is the AES.

Then the following key separation technique is suggested in [4]. Let K be a k-bit AES key. Then

$$\begin{cases} K_1 = \text{the first } k \text{ bits of } \mathrm{AES}_K(C_{1a}) \circ \mathrm{AES}_K(C_{1b}), \\ K_2 = \mathrm{AES}_K(C_2), \text{ and} \\ K_3 = \mathrm{AES}_K(C_3) \end{cases}$$

for some distinct constants C_{1a}, C_{1b}, C_2 and C_3. We call it XCBC+`kst` (key separation technique). XCBC+`kst` uses one k-bit key. However, it requires *additional* one key scheduling of AES and *additional* 3 or 4 AES invocations during the pre-processing time.

Similar discussion can be applied to TMAC. For example, we can let

$$\begin{cases} K_1 = \text{the first } k \text{ bits of } \mathrm{AES}_K(C_{1a}) \circ \mathrm{AES}_K(C_{1b}), \text{ and} \\ K_2 = \mathrm{AES}_K(C_2) \end{cases}$$

for some distinct constants C_{1a}, C_{1b} and C_2. We call it TMAC+`kst`.

We note that OMAC does *not* need such a key separation technique since its key length is k bits in its own form (without using any key separation technique). This saves storage space and pre-processing time compared to XCBC+`kst` and TMAC+`kst`.

Table 2. Efficiency comparison of CBC MAC and its variants.

Name	Domain	K len.	$\#K$ sche.	$\#E$ invo.	$\#E$ pre.
CBC MAC	$(\{0,1\}^n)^m$	k	1	$\|M\|/n$	0
EMAC	$(\{0,1\}^n)^+$	$2k$	2	$1+\|M\|/n$	0
RMAC	$\{0,1\}^*$	$2k$	$1+\#M$	$1+\lceil(\|M\|+1)/n\rceil$	0
XCBC	$\{0,1\}^*$	$k+2n$	1	$\lceil\|M\|/n\rceil$	0
TMAC	$\{0,1\}^*$	$k+n$	1	$\lceil\|M\|/n\rceil$	0
XCBC+`kst`	$\{0,1\}^*$	k	2	$\lceil\|M\|/n\rceil$	3 or 4
TMAC+`kst`	$\{0,1\}^*$	k	2	$\lceil\|M\|/n\rceil$	2 or 3
OMAC	$\{0,1\}^*$	k	1	$\lceil\|M\|/n\rceil$	1

A.3 Comparison

Let $E : \{0,1\}^k \times \{0,1\}^n \to \{0,1\}^n$ be a block cipher, and $M \in \{0,1\}^*$ be a message. We show an efficiency comparison of CBC MAC and its variants in Table 2, where:

- $(\{0,1\}^n)^+$ denotes the set of bit strings whose lengths are positive multiples of n.
- "K len." denotes the key length.
- "$\#K$ sche." denotes the number of block cipher key schedulings. For RMAC, it requires one block cipher key scheduling each time generating a tag.
- $\#M$ denotes the number messages which the sender has MACed.
- "$\#E$ invo." denotes the number of block cipher invocations to generate a tag for a message M, assuming $|M| > 0$.
- "$\#E$ pre." denotes the number of block cipher invocations during the pre-processing time. These block cipher invocations can be done without the message. For XCBC+`kst` and TMAC+`kst`, the block cipher is assumed to be the AES.

Next, let $E : \{0,1\}^k \times \{0,1\}^n \to \{0,1\}^n$ be the underlying block cipher used XCBC, TMAC and OMAC. In Table 3, we show a security comparison of XCBC, TMAC and OMAC. We see that there is no significant difference among them. They are equally secure up to the birthday paradox limit.

B Proof of Lemma 5.3

If A is a finite multiset then $\#A$ denotes the number of elements in A.

Let $\{a, b, c, \ldots\}$ be a finite multiset of bit strings. That is, $a \in \{0,1\}^*, b \in \{0,1\}^*, c \in \{0,1\}^*, \ldots$ hold. We say "$\{a, b, c, \ldots\}$ are distinct" if there exists no element occurs twice or more. Equivalently, $\{a, b, c, \ldots\}$ are distinct if any two elements in $\{a, b, c, \ldots\}$ are distinct.

Before proving Lemma 5.3, we need the following lemma.

Table 3. Security comparison of XCBC, TMAC and OMAC.

Name	Security Bound
XCBC [3, Corollary 2]	$\mathtt{Adv}^{\mathsf{mac}}_{\mathrm{XCBC}}(t,q,nm) \le \frac{(4m^2+1)q^2+1}{2^n} + 3 \cdot \mathtt{Adv}^{\mathsf{prp}}_E(t',q')$, where $t' = t + O(mq)$ and $q' = mq$.
TMAC [9, Theorem 5.1]	$\mathtt{Adv}^{\mathsf{mac}}_{\mathrm{TMAC}}(t,q,nm) \le \frac{(3m^2+1)q^2+1}{2^n} + \mathtt{Adv}^{\mathsf{prp}}_E(t',q')$, where $t' = t + O(mq)$ and $q' = mq$.
OMAC [Theorem 5.1]	$\mathtt{Adv}^{\mathsf{mac}}_{\mathrm{OMAC}}(t,q,nm) \le \frac{(5m^2+1)q^2+1}{2^n} + \mathtt{Adv}^{\mathsf{prp}}_E(t',q')$, where $t' = t + O(mq)$ and $q' = mq+1$.

Lemma B.1. *Let q_1,q_2,q_3,q_4,q_5,q_6 be six non-negative integers. For $1 \le i \le 6$, let $x_i^{(1)},\ldots,x_i^{(q_i)}$ be fixed n-bit strings such that $\{x_i^{(1)},\ldots,x_i^{(q_i)}\}$ are distinct. Similarly, for $1 \le i \le 6$, let $y_i^{(1)},\ldots,y_i^{(q_i)}$ be fixed n-bit strings such that*

- *$\{y_1^{(1)},\ldots,y_1^{(q_1)}\} \cup \{y_2^{(1)},\ldots,y_2^{(q_2)}\}$ are distinct, and*
- *$\{y_3^{(1)},\ldots,y_3^{(q_3)}\} \cup \{y_4^{(1)},\ldots,y_4^{(q_4)}\} \cup \{y_5^{(1)},\ldots,y_5^{(q_5)}\} \cup \{y_6^{(1)},\ldots,y_6^{(q_6)}\}$ are distinct.*

Let $P \in \mathit{Perm}(n)$ and $\mathtt{Rnd} \in \{0,1\}^n$. Then the number of $(P,\mathtt{Rnd})$ which satisfies

$$\begin{cases} Q_1(x_1^{(i)}) = y_1^{(i)} & \text{for } 1 \le {}^\forall i \le q_1, \\ Q_2(x_2^{(i)}) = y_2^{(i)} & \text{for } 1 \le {}^\forall i \le q_2, \\ Q_3(x_3^{(i)}) = y_3^{(i)} & \text{for } 1 \le {}^\forall i \le q_3, \\ Q_4(x_4^{(i)}) = y_4^{(i)} & \text{for } 1 \le {}^\forall i \le q_4, \\ Q_5(x_5^{(i)}) = y_5^{(i)} & \text{for } 1 \le {}^\forall i \le q_5 \text{ and} \\ Q_6(x_6^{(i)}) = y_6^{(i)} & \text{for } 1 \le {}^\forall i \le q_6 \end{cases} \tag{8}$$

is at least $(2^n - (q + q^2/2) \cdot (1 + \epsilon \cdot 2^n)) \cdot (2^n - q)!$, where $q = q_1 + \cdots + q_6$ and $\epsilon = \max\{\epsilon_1,\ldots,\epsilon_6\}$.

Proof. At the top level, we consider two cases: $\mathtt{Cst} \in \{x_1^{(1)},\ldots,x_1^{(q_1)}\}$ and $\mathtt{Cst} \notin \{x_1^{(1)},\ldots,x_1^{(q_1)}\}$.

Case 1: $\mathtt{Cst} \in \{x_1^{(1)},\ldots,x_1^{(q_1)}\}$. Let c be a unique integer such that $1 \le c \le q_1$ and $\mathtt{Cst} = x_1^{(c)}$. Let l be an n-bit variable. First, observe that:

$$\begin{aligned} &\#\{l \mid 1 \le {}^\exists i \le q_1, 1 \le {}^\exists j \le q_2, x_1^{(i)} = x_2^{(j)} \oplus y_1^{(c)} \oplus l\} \le q_1 q_2, \\ &\#\{l \mid 1 \le {}^\exists i \le q_1, 1 \le {}^\exists j \le q_3, x_1^{(i)} = x_3^{(j)} \oplus y_1^{(c)} \oplus l \oplus H_l(\mathtt{Cst}_1)\} \le q_1 q_3 \cdot \epsilon_4 \cdot 2^n, \\ &\#\{l \mid 1 \le {}^\exists i \le q_1, 1 \le {}^\exists j \le q_4, x_1^{(i)} = x_4^{(j)} \oplus y_1^{(c)} \oplus l \oplus H_l(\mathtt{Cst}_2)\} \le q_1 q_4 \cdot \epsilon_5 \cdot 2^n, \\ &\#\{l \mid 1 \le {}^\exists i \le q_1, 1 \le {}^\exists j \le q_5, x_1^{(i)} = x_5^{(j)} \oplus H_l(\mathtt{Cst}_1)\} \le q_1 q_5 \cdot \epsilon_1 \cdot 2^n, \\ &\#\{l \mid 1 \le {}^\exists i \le q_1, 1 \le {}^\exists j \le q_6, x_1^{(i)} = x_6^{(j)} \oplus H_l(\mathtt{Cst}_2)\} \le q_1 q_6 \cdot \epsilon_2 \cdot 2^n, \end{aligned}$$

$$
\begin{aligned}
&\#\{l \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_3, x_2^{(i)} = x_3^{(j)} \oplus H_l(\mathtt{Cst}_1)\} \le q_2q_3 \cdot \epsilon_1 \cdot 2^n,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_4, x_2^{(i)} = x_4^{(j)} \oplus H_l(\mathtt{Cst}_2)\} \le q_2q_4 \cdot \epsilon_2 \cdot 2^n,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_5, x_2^{(i)} \oplus y_1^{(c)} \oplus l = x_5^{(j)} \oplus H_l(\mathtt{Cst}_1)\} \le q_2q_5 \cdot \epsilon_4 \cdot 2^n,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_6, x_2^{(i)} \oplus y_1^{(c)} \oplus l = x_6^{(j)} \oplus H_l(\mathtt{Cst}_2)\} \le q_2q_6 \cdot \epsilon_5 \cdot 2^n,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_3, 1 \le {}^{\exists}j \le q_4, x_3^{(i)} \oplus H_l(\mathtt{Cst}_1) = x_4^{(j)} \oplus H_l(\mathtt{Cst}_2)\} \le q_3q_4 \cdot \epsilon_3 \cdot 2^n,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_3, 1 \le {}^{\exists}j \le q_5, x_3^{(i)} \oplus y_1^{(c)} \oplus l = x_5^{(j)}\} \le q_3q_5,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_3, 1 \le {}^{\exists}j \le q_6, x_3^{(i)} \oplus y_1^{(c)} \oplus l \oplus H_l(\mathtt{Cst}_1) = x_6^{(j)} \oplus H_l(\mathtt{Cst}_2)\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le q_3q_6 \cdot \epsilon_6 \cdot 2^n,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_4, 1 \le {}^{\exists}j \le q_5, x_4^{(i)} \oplus y_1^{(c)} \oplus l \oplus H_l(\mathtt{Cst}_2) = x_5^{(j)} \oplus H_l(\mathtt{Cst}_1)\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le q_4q_5 \cdot \epsilon_6 \cdot 2^n,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_4, 1 \le {}^{\exists}j \le q_6, x_4^{(i)} \oplus y_1^{(c)} \oplus l = x_6^{(j)}\} \le q_4q_6,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_5, 1 \le {}^{\exists}j \le q_6, x_5^{(i)} \oplus H_l(\mathtt{Cst}_1) = x_6^{(j)} \oplus H_l(\mathtt{Cst}_2)\} \le q_5q_6 \cdot \epsilon_3 \cdot 2^n,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_3, y_1^{(i)} \oplus y_1^{(c)} \oplus l = y_3^{(j)}\} \le q_1q_3,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_4, y_1^{(i)} \oplus y_1^{(c)} \oplus l = y_4^{(j)}\} \le q_1q_4,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_5, y_1^{(i)} \oplus y_1^{(c)} \oplus l = y_5^{(j)}\} \le q_1q_5,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_6, y_1^{(i)} \oplus y_1^{(c)} \oplus l = y_6^{(j)}\} \le q_1q_6,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_3, y_2^{(i)} \oplus y_1^{(c)} \oplus l = y_3^{(j)}\} \le q_2q_3,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_4, y_2^{(i)} \oplus y_1^{(c)} \oplus l = y_4^{(j)}\} \le q_2q_4,\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_5, y_2^{(i)} \oplus y_1^{(c)} \oplus l = y_5^{(j)}\} \le q_2q_5, \text{ and}\\
&\#\{l \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_6, y_2^{(i)} \oplus y_1^{(c)} \oplus l = y_6^{(j)}\} \le q_2q_6,
\end{aligned}
$$

from the conditions in Sec. 3.

We now fix any l which is *not* included in any of the above twenty-three sets. We have at least $(2^n - (q_1q_2 + q_1q_3 \cdot \epsilon_4 \cdot 2^n + q_1q_4 \cdot \epsilon_5 \cdot 2^n + q_1q_5 \cdot \epsilon_1 \cdot 2^n + q_1q_6 \cdot \epsilon_2 \cdot 2^n + q_2q_3 \cdot \epsilon_1 \cdot 2^n + q_2q_4 \cdot \epsilon_2 \cdot 2^n + q_2q_5 \cdot \epsilon_4 \cdot 2^n + q_2q_6 \cdot \epsilon_5 \cdot 2^n + q_3q_4 \cdot \epsilon_3 \cdot 2^n + q_3q_5 + q_3q_6 \cdot \epsilon_6 \cdot 2^n + q_4q_5 \cdot \epsilon_6 \cdot 2^n + q_4q_6 + q_5q_6 \cdot \epsilon_3 \cdot 2^n + q_1q_3 + q_1q_4 + q_1q_5 + q_1q_6 + q_2q_3 + q_2q_4 + q_2q_5 + q_2q_6)) \ge (2^n - q^2 \cdot \epsilon \cdot 2^n/2 - q^2/2)$ choice of such l.

Now we let $L \leftarrow l$ and $\mathtt{Rnd} \leftarrow l \oplus y_1^{(c)}$. Then we have:

- the inputs to P, $\{x_1^{(1)}, \ldots, x_1^{(q_1)}, x_2^{(1)} \oplus \mathtt{Rnd}, \ldots, x_2^{(q_2)} \oplus \mathtt{Rnd}, x_3^{(1)} \oplus \mathtt{Rnd} \oplus H_L(\mathtt{Cst}_1), \ldots, x_3^{(q_3)} \oplus \mathtt{Rnd} \oplus H_L(\mathtt{Cst}_1), x_4^{(1)} \oplus \mathtt{Rnd} \oplus H_L(\mathtt{Cst}_2), \ldots, x_4^{(q_4)} \oplus \mathtt{Rnd} \oplus H_L(\mathtt{Cst}_2), x_5^{(1)} \oplus H_L(\mathtt{Cst}_1), \ldots, x_5^{(q_5)} \oplus H_L(\mathtt{Cst}_1), x_6^{(1)} \oplus H_L(\mathtt{Cst}_2), \ldots, x_6^{(q_6)} \oplus H_L(\mathtt{Cst}_2)\}$, are distinct, and
- the corresponding outputs, $\{y_1^{(1)} \oplus \mathtt{Rnd}, \ldots, y_1^{(q_1)} \oplus \mathtt{Rnd}, y_2^{(1)} \oplus \mathtt{Rnd}, \ldots, y_2^{(q_2)} \oplus \mathtt{Rnd}, y_3^{(1)}, \ldots, y_3^{(q_3)}, y_4^{(1)}, \ldots, y_4^{(q_4)}, y_5^{(1)}, \ldots, y_5^{(q_5)}, y_6^{(1)}, \ldots, y_6^{(q_6)}\}$, are distinct.

In other words, for P, the above $q_1 + q_2 + q_3 + q_4 + q_5 + q_6$ input-output pairs are determined. The remaining $2^n - (q_1 + q_2 + q_3 + q_4 + q_5 + q_6)$ input-output pairs are undetermined. Therefore we have $(2^n - (q_1 + q_2 + q_3 + q_4 + q_5 + q_6))! = (2^n - q)!$ possible choice of P for any such fixed $(L, \mathtt{Rnd})$.

Case 2: $\texttt{Cst} \notin \{x_1^{(1)}, \ldots, x_1^{(q_1)}\}$. In this case, we count the number of $\texttt{Rnd}$ and L independently. Then similar to Case 1, observe that:

$$\begin{aligned}
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_2, \texttt{Cst} = x_2^{(i)} \oplus \texttt{Rnd}\} \le q_2,\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_2, x_1^{(i)} = x_2^{(j)} \oplus \texttt{Rnd}\} \le q_1q_2,\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_3, 1 \le {}^{\exists}j \le q_5, x_3^{(i)} \oplus \texttt{Rnd} = x_5^{(j)}\} \le q_3q_5,\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_4, 1 \le {}^{\exists}j \le q_6, x_4^{(i)} \oplus \texttt{Rnd} = x_6^{(j)}\} \le q_4q_6,\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_3, y_1^{(i)} \oplus \texttt{Rnd} = y_3^{(j)}\} \le q_1q_3,\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_4, y_1^{(i)} \oplus \texttt{Rnd} = y_4^{(j)}\} \le q_1q_4,\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_5, y_1^{(i)} \oplus \texttt{Rnd} = y_5^{(j)}\} \le q_1q_5,\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_6, y_1^{(i)} \oplus \texttt{Rnd} = y_6^{(j)}\} \le q_1q_6,\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_3, y_2^{(i)} \oplus \texttt{Rnd} = y_3^{(j)}\} \le q_2q_3,\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_4, y_2^{(i)} \oplus \texttt{Rnd} = y_4^{(j)}\} \le q_2q_4,\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_5, y_2^{(i)} \oplus \texttt{Rnd} = y_5^{(j)}\} \le q_2q_5, \text{ and}\\
&\#\{\texttt{Rnd} \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_6, y_2^{(i)} \oplus \texttt{Rnd} = y_6^{(j)}\} \le q_2q_6.
\end{aligned}$$

We fix any $\texttt{Rnd}$ which is *not* included in any of the above twelve sets. We have at least $(2^n - (q_2 + q_1q_2 + q_3q_5 + q_4q_6 + q_1q_3 + q_1q_4 + q_1q_5 + q_1q_6 + q_2q_3 + q_2q_4 + q_2q_5 + q_2q_6)) \ge (2^n - q - q^2/2)$ choice of such $\texttt{Rnd}$.

Next we see that:

$$\begin{aligned}
&\#\{L \mid 1 \le {}^{\exists}i \le q_3, \texttt{Cst} = x_3^{(i)} \oplus \texttt{Rnd} \oplus H_L(\texttt{Cst}_1)\} \le q_3 \cdot \epsilon_1 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_4, \texttt{Cst} = x_4^{(i)} \oplus \texttt{Rnd} \oplus H_L(\texttt{Cst}_2)\} \le q_4 \cdot \epsilon_2 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_5, \texttt{Cst} = x_5^{(i)} \oplus H_L(\texttt{Cst}_1)\} \le q_5 \cdot \epsilon_1 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_6, \texttt{Cst} = x_6^{(i)} \oplus H_L(\texttt{Cst}_2)\} \le q_6 \cdot \epsilon_2 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_3, x_1^{(i)} = x_3^{(j)} \oplus \texttt{Rnd} \oplus H_L(\texttt{Cst}_1)\} \le q_1q_3 \cdot \epsilon_1 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_4, x_1^{(i)} = x_4^{(j)} \oplus \texttt{Rnd} \oplus H_L(\texttt{Cst}_2)\} \le q_1q_4 \cdot \epsilon_2 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_5, x_1^{(i)} = x_5^{(j)} \oplus H_L(\texttt{Cst}_1)\} \le q_1q_5 \cdot \epsilon_1 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_1, 1 \le {}^{\exists}j \le q_6, x_1^{(i)} = x_6^{(j)} \oplus H_L(\texttt{Cst}_2)\} \le q_1q_6 \cdot \epsilon_2 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_3, x_2^{(i)} = x_3^{(j)} \oplus H_L(\texttt{Cst}_1)\} \le q_2q_3 \cdot \epsilon_1 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_4, x_2^{(i)} = x_4^{(j)} \oplus H_L(\texttt{Cst}_2)\} \le q_2q_4 \cdot \epsilon_2 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_5, x_2^{(i)} \oplus \texttt{Rnd} = x_5^{(j)} \oplus H_L(\texttt{Cst}_1)\} \le q_2q_5 \cdot \epsilon_1 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_2, 1 \le {}^{\exists}j \le q_6, x_2^{(i)} \oplus \texttt{Rnd} = x_6^{(j)} \oplus H_L(\texttt{Cst}_2)\} \le q_2q_6 \cdot \epsilon_2 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_3, 1 \le {}^{\exists}j \le q_4, x_3^{(i)} \oplus H_L(\texttt{Cst}_1) = x_4^{(j)} \oplus H_L(\texttt{Cst}_2)\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le q_3q_4 \cdot \epsilon_3 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_3, 1 \le {}^{\exists}j \le q_6, x_3^{(i)} \oplus \texttt{Rnd} \oplus H_L(\texttt{Cst}_1) = x_6^{(j)} \oplus H_L(\texttt{Cst}_2)\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le q_3q_6 \cdot \epsilon_3 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_4, 1 \le {}^{\exists}j \le q_5, x_4^{(i)} \oplus \texttt{Rnd} \oplus H_L(\texttt{Cst}_2) = x_5^{(j)} \oplus H_L(\texttt{Cst}_1)\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le q_4q_5 \cdot \epsilon_3 \cdot 2^n,\\
&\#\{L \mid 1 \le {}^{\exists}i \le q_5, 1 \le {}^{\exists}j \le q_6, x_5^{(i)} \oplus H_L(\texttt{Cst}_1) = x_6^{(j)} \oplus H_L(\texttt{Cst}_2)\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le q_5q_6 \cdot \epsilon_3 \cdot 2^n,
\end{aligned}$$

$$\#\{L \mid 1 \le {}^\exists i \le q_1, L = y_1^{(i)} \oplus \mathtt{Rnd}\} \le q_1,$$
$$\#\{L \mid 1 \le {}^\exists i \le q_2, L = y_2^{(i)} \oplus \mathtt{Rnd}\} \le q_2,$$
$$\#\{L \mid 1 \le {}^\exists i \le q_3, L = y_3^{(i)}\} \le q_3,$$
$$\#\{L \mid 1 \le {}^\exists i \le q_4, L = y_4^{(i)}\} \le q_4,$$
$$\#\{L \mid 1 \le {}^\exists i \le q_5, L = y_5^{(i)}\} \le q_5, \text{ and}$$
$$\#\{L \mid 1 \le {}^\exists i \le q_6, L = y_6^{(i)}\} \le q_6,$$

from the conditions in Sec. 3.

We now fix any L which is *not* included in any of the above twenty-two sets. We have at least $(2^n - (q_3 \cdot \epsilon_1 \cdot 2^n + q_4 \cdot \epsilon_2 \cdot 2^n + q_5 \cdot \epsilon_1 \cdot 2^n + q_6 \cdot \epsilon_2 \cdot 2^n + q_1 q_3 \cdot \epsilon_1 \cdot 2^n + q_1 q_4 \cdot \epsilon_2 \cdot 2^n + q_1 q_5 \cdot \epsilon_1 \cdot 2^n + q_1 q_6 \cdot \epsilon_2 \cdot 2^n + q_2 q_3 \cdot \epsilon_1 \cdot 2^n + q_2 q_4 \cdot \epsilon_2 \cdot 2^n + q_2 q_5 \cdot \epsilon_1 \cdot 2^n + q_2 q_6 \cdot \epsilon_2 \cdot 2^n + q_3 q_4 \cdot \epsilon_3 \cdot 2^n + q_3 q_6 \cdot \epsilon_3 \cdot 2^n + q_4 q_5 \cdot \epsilon_3 \cdot 2^n + q_5 q_6 \cdot \epsilon_3 \cdot 2^n + q_1 + q_2 + q_3 + q_4 + q_5 + q_6)) \ge (2^n - q \cdot \epsilon \cdot 2^n - q^2 \cdot \epsilon \cdot 2^n/2 - q)$ choice of such L.

Then we have:

- the inputs to P, $\{\mathtt{Cst}, x_1^{(1)}, \dots, x_1^{(q_1)}, x_2^{(1)} \oplus \mathtt{Rnd}, \dots, x_2^{(q_2)} \oplus \mathtt{Rnd}, x_3^{(1)} \oplus \mathtt{Rnd} \oplus H_L(\mathtt{Cst}_1), \dots, x_3^{(q_3)} \oplus \mathtt{Rnd} \oplus H_L(\mathtt{Cst}_1), x_4^{(1)} \oplus \mathtt{Rnd} \oplus H_L(\mathtt{Cst}_2), \dots, x_4^{(q_4)} \oplus \mathtt{Rnd} \oplus H_L(\mathtt{Cst}_2), x_5^{(1)} \oplus H_L(\mathtt{Cst}_1), \dots, x_5^{(q_5)} \oplus H_L(\mathtt{Cst}_1), x_6^{(1)} \oplus H_L(\mathtt{Cst}_2), \dots, x_6^{(q_6)} \oplus H_L(\mathtt{Cst}_2)\}$, are distinct, and
- the corresponding outputs, $\{L, y_1^{(1)} \oplus \mathtt{Rnd}, \dots, y_1^{(q_1)} \oplus \mathtt{Rnd}, y_2^{(1)} \oplus \mathtt{Rnd}, \dots, y_2^{(q_2)} \oplus \mathtt{Rnd}, y_3^{(1)}, \dots, y_3^{(q_3)}, y_4^{(1)}, \dots, y_4^{(q_4)}, y_5^{(1)}, \dots, y_5^{(q_5)}, y_6^{(1)}, \dots, y_6^{(q_6)}\}$, are distinct.

In other words, for P, the above $1 + q_1 + q_2 + q_3 + q_4 + q_5 + q_6$ input-output pairs are determined. The remaining $2^n - (1 + q_1 + q_2 + q_3 + q_4 + q_5 + q_6)$ input-output pairs are undetermined. Therefore we have $(2^n - (1 + q_1 + q_2 + q_3 + q_4 + q_5 + q_6))! = (2^n - (1 + q))!$ possible choice of P for any such fixed $(L, \mathtt{Rnd})$.

Completing the Proof. In Case 1, we have at least $(2^n - (q^2/2) \cdot (1 + \epsilon \cdot 2^n)) \cdot (2^n - q)!$ choice of $(P, \mathtt{Rnd})$ which satisfies (8).

In Case 2, we have at least $(2^n - q - q^2/2) \cdot (2^n - q \cdot \epsilon \cdot 2^n - q^2 \cdot \epsilon \cdot 2^n/2 - q) \cdot (2^n - (1 + q))!$ choice of $(P, \mathtt{Rnd})$ which satisfies (8). This bound is at least $(2^n - (q + q^2/2) \cdot (1 + \epsilon \cdot 2^n)) \cdot (2^n - q)!$.

This conclude s the proof of the lemma. □

We now prove Lemma 5.3.

Proof (of Lemma 5.3). For $1 \le i \le 6$, let $\mathcal{O}_i$ be either Q_i or P_i. The adversary $\mathcal{A}$ has oracle access to $\mathcal{O}_1, \dots, \mathcal{O}_6$. Since $\mathcal{A}$ is computationally unbounded, there is no loss of generality to assume that $\mathcal{A}$ is deterministic.

There are six types of queries $\mathcal{A}$ can make: $(\mathcal{O}_j, x)$ which denotes the query "what is $\mathcal{O}_j(x)$?" For the i-th query $\mathcal{A}$ makes to $\mathcal{O}_j$, define the query-answer pair $(x_j^{(i)}, y_j^{(i)}) \in \{0,1\}^n \times \{0,1\}^n$, where $\mathcal{A}$'s query was $(\mathcal{O}_j, x_j^{(i)})$ and the answer it got was $y_j^{(i)}$.

Suppose that we run $\mathcal{A}$ with oracles $\mathcal{O}_1, \dots, \mathcal{O}_6$. For this run, assume that $\mathcal{A}$ made q_j queries to $\mathcal{O}_j(\cdot)$, where $q_1 + \cdots + q_6 = q$. For this run, we define view v

of $\mathcal{A}$ as

$$v \stackrel{\text{def}}{=} \langle (y_1^{(1)}, \ldots, y_1^{(q_1)}), (y_2^{(1)}, \ldots, y_2^{(q_2)}), (y_3^{(1)}, \ldots, y_3^{(q_3)}), \\ (y_4^{(1)}, \ldots, y_4^{(q_4)}), (y_5^{(1)}, \ldots, y_5^{(q_5)}), (y_6^{(1)}, \ldots, y_6^{(q_6)}) \rangle \ . \tag{9}$$

For this view, we always have:

$$\text{For } 1 \leq j \leq 6,\ \{y_j^{(1)}, \ldots, y_j^{(q_j)}\} \text{ are distinct.}$$

We note that since $\mathcal{A}$ never repeats a query, for the corresponding queries, we have:

$$\text{For } 1 \leq j \leq 6,\ \{x_j^{(1)}, \ldots, x_j^{(q_j)}\} \text{ are distinct.}$$

Since $\mathcal{A}$ is deterministic, the i-th query $\mathcal{A}$ makes is fully determined by the first $i-1$ query-answer pairs. This implies that if we fix some qn-bit string V and return the i-th n-bit block as the answer for the i-th query $\mathcal{A}$ makes (instead of the oracles), then

- $\mathcal{A}$'s queries are uniquely determined,
- $q_1, \ldots, q_6$ are uniquely determined,
- the parsing of V into the format defined in (9) is uniquely determined, and
- the final output of $\mathcal{A}$ (0 or 1) is uniquely determined.

Let $\boldsymbol{V}_{one}$ be a set of all qn-bit strings V such that $\mathcal{A}$ outputs 1. We let $N_{one} \stackrel{\text{def}}{=} \#\boldsymbol{V}_{one}$. Also, let $\boldsymbol{V}_{good}$ be a set of all qn-bit strings V such that:

$$\text{For } 1 \leq {}^{\forall} i < {}^{\forall} j \leq q, \text{ the } i\text{-th } n\text{-bit block of } V \neq \text{the } j\text{-th } n\text{-bit block of } V.$$

Note that if $V \in \boldsymbol{V}_{good}$ then the corresponding parsing v satisfies:

- $\{y_1^{(1)}, \ldots, y_1^{(q_1)}\} \cup \{y_2^{(1)}, \ldots, y_2^{(q_2)}\}$ are distinct, and
- $\{y_3^{(1)}, \ldots, y_3^{(q_3)}\} \cup \{y_4^{(1)}, \ldots, y_4^{(q_4)}\} \cup \{y_5^{(1)}, \ldots, y_5^{(q_5)}\} \cup \{y_6^{(1)}, \ldots, y_6^{(q_6)}\}$ are distinct.

Now observe that the number of V which is *not* in the set $\boldsymbol{V}_{good}$ is at most $\binom{q}{2} \frac{2^{qn}}{2^n}$. Therefore, we have

$$\#\{V \mid V \in (\boldsymbol{V}_{one} \cap \boldsymbol{V}_{good})\} \geq N_{one} - \binom{q}{2} \frac{2^{qn}}{2^n} \ . \tag{10}$$

Evaluation of p_{rand}. We first evaluate

$$p_{rand} \stackrel{\text{def}}{=} \Pr(P_1, \ldots, P_6 \stackrel{R}{\leftarrow} \mathrm{Perm}(n) : \mathcal{A}^{P_1(\cdot), \ldots, P_6(\cdot)} = 1) \\ = \frac{\#\{(P_1, \ldots, P_6) \mid \mathcal{A}^{P_1(\cdot), \ldots, P_6(\cdot)} = 1\}}{\{(2^n)!\}^6} \ .$$

For each $V \in \boldsymbol{V}_{one}$, the number of $(P_1, \ldots, P_6)$ such that

$$\text{For } 1 \leq j \leq 6,\ P_j(x_j^{(i)}) = y_j^{(i)} \text{ for } 1 \leq {}^{\forall} i \leq q_j, \tag{11}$$

is exactly $\prod_{1\le j\le 6}(2^n-q_j)!$, which is at most $(2^n-q)!\cdot\{(2^n)!\}^5$. Therefore, we have

$$\begin{aligned}
p_{rand} &= \sum_{V\in\boldsymbol{V}_{one}} \frac{\#\{(P_1,\ldots,P_6)\mid (P_1,\ldots,P_6) \text{ satisfying (11)}\}}{\{(2^n)!\}^6}\\
&\le \sum_{V\in\boldsymbol{V}_{one}} \frac{(2^n-q)!}{(2^n)!}\\
&= N_{one}\cdot\frac{(2^n-q)!}{(2^n)!} \quad .
\end{aligned}$$

Evaluation of p_{real}. We next evaluate

$$\begin{aligned}
p_{real} &\stackrel{\text{def}}{=} \Pr(P \stackrel{R}{\leftarrow} \mathrm{Perm}(n); \mathtt{Rnd} \stackrel{R}{\leftarrow} \{0,1\}^n : \mathcal{A}^{Q_1(\cdot),\ldots,Q_6(\cdot)}=1)\\
&= \frac{\#\{(P,\mathtt{Rnd})\mid \mathcal{A}^{Q_1(\cdot),\ldots,Q_6(\cdot)}=1\}}{(2^n)!\cdot 2^n} \quad .
\end{aligned}$$

Then from Lemma B.1, we have

$$\begin{aligned}
p_{real} &\ge \sum_{V\in(\boldsymbol{V}_{one}\cap\boldsymbol{V}_{good})} \frac{\#\,\{(P,\mathtt{Rnd})\mid (P,\mathtt{Rnd}) \text{ satisfying (8)}\}}{(2^n)!\cdot 2^n}\\
&\ge \sum_{V\in(\boldsymbol{V}_{one}\cap\boldsymbol{V}_{good})} \frac{(2^n-q)!}{(2^n)!}\cdot\left(1-\frac{(q+q^2/2)\cdot(1+\epsilon\cdot 2^n)}{2^n}\right) \quad .
\end{aligned}$$

Completing the Proof. From (10) we have

$$\begin{aligned}
p_{real} &\ge \left(N_{one}-\binom{q}{2}\frac{2^{qn}}{2^n}\right)\cdot\frac{(2^n-q)!}{(2^n)!}\cdot\left(1-\frac{(q+q^2/2)\cdot(1+\epsilon\cdot 2^n)}{2^n}\right)\\
&\ge \left(p_{rand}-\binom{q}{2}\frac{2^{qn}}{2^n}\cdot\frac{(2^n-q)!}{(2^n)!}\right)\cdot\left(1-\frac{(q+q^2/2)\cdot(1+\epsilon\cdot 2^n)}{2^n}\right) \quad .
\end{aligned}$$

Since $2^{qn}\cdot\frac{(2^n-q)!}{(2^n)!}\ge 1$, we have

$$\begin{aligned}
p_{real} &\ge \left(p_{rand}-\frac{q(q-1)}{2\cdot 2^n}\right)\cdot\left(1-\frac{(q+q^2/2)\cdot(1+\epsilon\cdot 2^n)}{2^n}\right)\\
&\ge p_{rand}-\frac{(2q^2+q)+(q^2+2q)\cdot\epsilon\cdot 2^n}{2\cdot 2^n}\\
&\ge p_{rand}-\frac{3q^2}{2}\cdot\left(\frac{1}{2^n}+\epsilon\right) \quad . \qquad (12)
\end{aligned}$$

Applying the same argument to $1-p_{real}$ and $1-p_{rand}$ yields that

$$1-p_{real}\ge 1-p_{rand}-\frac{3q^2}{2}\cdot\left(\frac{1}{2^n}+\epsilon\right) \quad . \qquad (13)$$

Finally, (12) and (13) give $|p_{real}-p_{rand}|\le\frac{3q^2}{2}\cdot\left(\frac{1}{2^n}+\epsilon\right)$. □

C Proof of Lemma 5.4

Let S and S' be distinct bit strings such that $|S| = sn$ for some $s \geq 1$, and $|S'| = s'n$ for some $s' \geq 1$. Define $V_n(S, S') \stackrel{\text{def}}{=} \Pr(P_2 \stackrel{R}{\leftarrow} \text{Perm}(n) : \text{CBC}_{P_2}(S) = \text{CBC}_{P_2}(S'))$. Then the following proposition is known [3].

Proposition C.1 (Black and Rogaway [3]). *Let S and S' be distinct bit strings such that $|S| = sn$ for some $s \geq 1$, and $|S'| = s'n$ for some $s' \geq 1$. Assume that $s, s' \leq 2^n/4$. Then*

$$V_n(S, S') \leq \frac{(s + s')^2}{2^n} \quad .$$

Now let M and M' be distinct bit strings such that $|M| = mn$ for some $m \geq 2$, and $|M'| = m'n$ for some $m' \geq 2$. Define $W_n(M, M') \stackrel{\text{def}}{=} \Pr(P_1, \ldots, P_6 \stackrel{R}{\leftarrow} \text{Perm}(n) : \text{MOMAC}_{P_1,\ldots,P_6}(M) = \text{MOMAC}_{P_1,\ldots,P_6}(M'))$. We note that P_5 and P_6 are irrelevant in the event $\text{MOMAC}_{P_1,\ldots,P_6}(M) = \text{MOMAC}_{P_1,\ldots,P_6}(M')$ since M and M' are both longer than n bits. Also, P_4 is irrelevant in the above event since $|M|$ and $|M'|$ are both multiples of n. Further, P_3 is irrelevant in the above event since it is invertible, and thus, there is a collision if and only if there is a collision at the input to the last encryption.

We show the following lemma.

Lemma C.1 (MOMAC Collision Bound). *Let M and M' be distinct bit strings such that $|M| = mn$ for some $m \geq 2$, and $|M'| = m'n$ for some $m' \geq 2$. Assume that $m, m' \leq 2^n/4$. Then*

$$W_n(M, M') \leq \frac{(m + m')^2}{2^n} \quad .$$

Proof. Let $M[1] \cdots M[m]$ and $M'[1] \cdots M'[m']$ be partitions of M and M' respectively. We consider two cases: $M[1] = M'[1]$ and $M[1] \neq M'[1]$.

Case 1: $M[1] = M'[1]$. In this case, Let P_1 be any permutation in $\text{Perm}(n)$, and let $S \leftarrow (P_1(M[1]) \oplus M[2]) \circ M[3] \circ \cdots \circ M[m]$ and $S' \leftarrow (P_1(M'[1]) \oplus M'[2]) \circ M'[3] \circ \cdots \circ M'[m']$. Observe that $\text{MOMAC}_{P_1,\ldots,P_6}(M) = \text{MOMAC}_{P_1,\ldots,P_6}(M')$ if and only if $\text{CBC}_{P_2}(S) = \text{CBC}_{P_2}(S')$, since we may ignore the last encryptions in $\text{CBC}_{P_2}(S)$ and $\text{CBC}_{P_2}(S')$. Therefore

$$W_n(M, M') \leq V_n(S, S') \leq \frac{(m + m' - 2)^2}{2^n} \quad .$$

Case 2: $M[1] \neq M'[1]$. In this case, we split into two cases: $P_1(M[1]) \oplus M[2] \neq P_1(M'[1]) \oplus M'[2]$ and $P_1(M[1]) \oplus M[2] = P_1(M'[1]) \oplus M'[2]$. The former event will occur with probability at most 1. The later one will occur with probability at most $\frac{1}{2^n - 1}$, which is at most $\frac{2}{2^n}$. Then it is not hard to see that

$$W_n(M, M') \leq 1 \cdot V_n(S, S') + \frac{2}{2^n} \leq \frac{(m + m - 2)^2}{2^n} + \frac{2}{2^n} \leq \frac{(m + m')^2}{2^n}$$

by applying the similar argument as in Case 1. $\square$

Let m be an integer such that $m \le 2^n/4$. We consider the following four sets.

$$\begin{cases} D_1 \stackrel{\text{def}}{=} \{M \mid M \in \{0,1\}^*,\, n < |M| \le mn \text{ and } |M| \text{ is a multiple of } n\} \\ D_2 \stackrel{\text{def}}{=} \{M \mid M \in \{0,1\}^*,\, n < |M| \le mn \text{ and } |M| \text{ is not a multiple of } n\} \\ D_3 \stackrel{\text{def}}{=} \{M \mid M \in \{0,1\}^* \text{ and } |M| = n\} \\ D_4 \stackrel{\text{def}}{=} \{M \mid M \in \{0,1\}^* \text{ and } |M| < n\} \end{cases}$$

We next show the following lemma.

Lemma C.2. *Let q_1, q_2, q_3, q_4 be four non-negative integers. For $1 \le i \le 4$, let $M_i^{(1)}, \ldots, M_i^{(q_i)}$ be fixed bit strings such that $M_i^{(j)} \in D_i$ for $1 \le j \le q_i$ and $\{M_i^{(1)}, \ldots, M_i^{(q_i)}\}$ are distinct. Similarly, for $1 \le i \le 4$, let $T_i^{(1)}, \ldots, T_i^{(q_i)}$ be fixed n-bit strings such that $\{T_i^{(1)}, \ldots, T_i^{(q_i)}\}$ are distinct. Then the number of $P_1, \ldots, P_6 \in Perm(n)$ such that*

$$\begin{cases} MOMAC_{P_1,\ldots,P_6}(M_1^{(i)}) = T_1^{(i)} \text{ for } 1 \le {}^\forall i \le q_1, \\ MOMAC_{P_1,\ldots,P_6}(M_2^{(i)}) = T_2^{(i)} \text{ for } 1 \le {}^\forall i \le q_2, \\ MOMAC_{P_1,\ldots,P_6}(M_3^{(i)}) = T_3^{(i)} \text{ for } 1 \le {}^\forall i \le q_3 \text{ and} \\ MOMAC_{P_1,\ldots,P_6}(M_4^{(i)}) = T_4^{(i)} \text{ for } 1 \le {}^\forall i \le q_4 \end{cases} \tag{14}$$

is at least $\{(2^n)!\}^6 \left(1 - \frac{2q^2m^2}{2^n}\right) \cdot \frac{1}{2^{qn}}$, where $q = q_1 + \cdots + q_4$.

Proof. We first consider $M_1^{(1)}, \ldots, M_1^{(q_1)}$. The number of (P_1, P_2) such that

$$\mathrm{MOMAC}_{P_1,\ldots,P_6}(M_1^{(i)}) = \mathrm{MOMAC}_{P_1,\ldots,P_6}(M_1^{(j)}) \text{ for } 1 \le {}^\exists i < {}^\exists j \le q_1$$

is at most $\{(2^n)!\}^2 \cdot \binom{q_1}{2} \cdot \frac{4m^2}{2^n}$ from Lemma C.1. Note that $P_3, \ldots, P_6$ are irrelevant in the above event.

We next consider $M_2^{(1)}, \ldots, M_2^{(q_2)}$. The number of (P_1, P_2) such that

$$\mathrm{MOMAC}_{P_1,\ldots,P_6}(M_2^{(i)}) = \mathrm{MOMAC}_{P_1,\ldots,P_6}(M_2^{(j)}) \text{ for } 1 \le {}^\exists i < {}^\exists j \le q_2$$

is at most $\{(2^n)!\}^2 \cdot \binom{q_2}{2} \cdot \frac{4m^2}{2^n}$ from Lemma C.1.

Now we fix any (P_1, P_2) which is not like the above. We have at least $\{(2^n)!\}^2 \left(1 - \binom{q_1}{2} \cdot \frac{4m^2}{2^n} - \binom{q_2}{2} \cdot \frac{4m^2}{2^n}\right)$ choice.

Now P_1 and P_2 are fixed in such a way that the inputs to P_3 are distinct and the inputs to P_4 are distinct. Also, the corresponding outputs $\{T_3^{(1)}, \ldots, T_3^{(q_3)}\}$ are distinct, and $\{T_4^{(1)}, \ldots, T_4^{(q_4)}\}$ are distinct. We know that the inputs to P_5 are distinct, and the corresponding outputs $\{T_3^{(1)}, \ldots, T_3^{(q_3)}\}$ are distinct. Also, the inputs to P_6 are distinct, and and the corresponding outputs $\{T_4^{(1)}, \ldots, T_4^{(q_4)}\}$ are distinct. Therefore, we have at least $\{(2^n)!\}^2 \left(1 - \binom{q_1}{2} \cdot \frac{4m^2}{2^n} - \binom{q_2}{2} \cdot \frac{4m^2}{2^n}\right) \cdot (2^n - q_1)! \cdot (2^n - q_2)! \cdot (2^n - q_3)! \cdot (2^n - q_4)!$ choice of $P_1, \ldots, P_6$ which satisfies (14). This bound is at least $\{(2^n)!\}^6 \left(1 - \frac{2q^2m^2}{2^n}\right) \cdot \frac{1}{2^{qn}}$ since $(2^n - q_i)! \ge \frac{(2^n)!}{2^{q_i n}}$.

This concludes the proof of the lemma. $\square$

We now prove Lemma 5.4.

Proof (of Lemma 5.4). Let $\mathcal{O}$ be either $\mathrm{MOMAC}_{P_1,\ldots,P_6}$ or R. Since $\mathcal{A}$ is computationally unbounded, there is no loss of generality to assume that $\mathcal{A}$ is deterministic.

Similar to the proof of Lemma 5.3, for the query $\mathcal{A}$ makes to the oracle $\mathcal{O}$, define the query-answer pair $(M_j^{(i)}, T_j^{(i)}) \in D_j \times \{0,1\}^n$, where $\mathcal{A}$'s i-th query in D_j was $M_j^{(i)} \in D_j$ and the answer it got was $T_j^{(i)} \in \{0,1\}^n$.

Suppose that we run $\mathcal{A}$ with the oracle. For this run, assume that $\mathcal{A}$ made q_j queries in D_j, where $1 \le j \le 4$ and $q_1 + \cdots + q_4 = q$. For this run, we define view v of $\mathcal{A}$ as

$$\begin{aligned} v \stackrel{\text{def}}{=} \langle (T_1^{(1)}, \ldots, T_1^{(q_1)}), (T_2^{(1)}, \ldots, T_2^{(q_2)}), \\ (T_3^{(1)}, \ldots, T_3^{(q_3)}), (T_4^{(1)}, \ldots, T_4^{(q_4)}) \rangle \ . \end{aligned} \tag{15}$$

Since $\mathcal{A}$ is deterministic, the i-th query $\mathcal{A}$ makes is fully determined by the first $i-1$ query-answer pairs. This implies that if we fix some qn-bit string V and return the i-th n-bit block as the answer for the i-th query $\mathcal{A}$ makes (instead of the oracle), then

- $\mathcal{A}$'s queries are uniquely determined,
- $q_1, \ldots, q_4$ are uniquely determined,
- the parsing of V into the format defined in (15) is uniquely determined, and
- the final output of $\mathcal{A}$ (0 or 1) is uniquely determined.

Let $\boldsymbol{V}_{one}$ be a set of all qn-bit strings V such that $\mathcal{A}$ outputs 1. We let $N_{one} \stackrel{\text{def}}{=} \#\boldsymbol{V}_{one}$. Also, let $\boldsymbol{V}_{good}$ be a set of all qn-bit strings V such that:

For $1 \le {}^\forall i < {}^\forall j \le q$, the i-th n-bit block of $V \ne$ the j-th n-bit block of V.

Note that if $V \in \boldsymbol{V}_{good}$, then the corresponding parsing v of V satisfies that: $\{T_1^{(1)}, \ldots, T_1^{(q_1)}\}$ are distinct, $\{T_2^{(1)}, \ldots, T_2^{(q_2)}\}$ are distinct, $\{T_3^{(1)}, \ldots, T_3^{(q_3)}\}$ are distinct and $\{T_4^{(1)}, \ldots, T_4^{(q_4)}\}$ are distinct. Now observe that the number of V which is *not* in the set $\boldsymbol{V}_{good}$ is at most $\binom{q}{2}\frac{2^{qn}}{2^n}$. Therefore, we have

$$\#\{V \mid V \in (\boldsymbol{V}_{one} \cap \boldsymbol{V}_{good})\} \ge N_{one} - \binom{q}{2}\frac{2^{qn}}{2^n} \ . \tag{16}$$

Evaluation of p_{rand}. We first evaluate

$$p_{rand} \stackrel{\text{def}}{=} \Pr(R \stackrel{R}{\leftarrow} \mathrm{Rand}(*, n) : \mathcal{A}^{R(\cdot)} = 1) \ .$$

Then it is not hard to see

$$p_{rand} = \sum_{V \in \boldsymbol{V}_{one}} \frac{1}{2^{qn}} = \frac{N_{one}}{2^{qn}} \ .$$

Evaluation of p_{real}. We next evaluate

$$\begin{aligned} p_{real} &\stackrel{\text{def}}{=} \Pr(P_1, \ldots, P_6 \stackrel{R}{\leftarrow} \mathrm{Perm}(n) : \mathcal{A}^{\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)} = 1) \\ &= \frac{\#\{(P_1, \ldots, P_6) \mid \mathcal{A}^{\mathrm{MOMAC}_{P_1,\ldots,P_6}(\cdot)} = 1\}}{\{(2^n)!\}^6} \ . \end{aligned}$$

Then from Lemma C.2, we have

$$\begin{aligned} p_{real} &\geq \sum_{V \in (\boldsymbol{V}_{one} \cap \boldsymbol{V}_{good})} \frac{\#\{(P_1, \ldots, P_6) \mid (P_1, \ldots, P_6) \text{ satisfying (14)}\}}{\{(2^n)!\}^6} \\ &\geq \sum_{V \in (\boldsymbol{V}_{one} \cap \boldsymbol{V}_{good})} \left(1 - \frac{2q^2m^2}{2^n}\right) \cdot \frac{1}{2^{qn}} \ . \end{aligned}$$

Completing the Proof. From (16) we have

$$\begin{aligned} p_{real} &\geq \left(N_{one} - \binom{q}{2}\frac{2^{qn}}{2^n}\right) \cdot \left(1 - \frac{2q^2m^2}{2^n}\right) \cdot \frac{1}{2^{qn}} \\ &= \left(p_{rand} - \binom{q}{2}\frac{1}{2^n}\right) \cdot \left(1 - \frac{2q^2m^2}{2^n}\right) \\ &\geq p_{rand} - \binom{q}{2}\frac{1}{2^n} - \frac{2q^2m^2}{2^n} \\ &\geq p_{rand} - \frac{2q^2m^2 + q^2}{2^n} \ . \end{aligned} \tag{17}$$

Applying the same argument to $1 - p_{real}$ and $1 - p_{rand}$ yields that

$$1 - p_{real} \geq 1 - p_{rand} - \frac{2q^2m^2 + q^2}{2^n} \ . \tag{18}$$

Finally, (17) and (18) give $|p_{real} - p_{rand}| \leq \frac{2q^2m^2+q^2}{2^n}$. □