Which Languages Have 4-Round Zero-Knowledge Proofs?

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Abstract. We show that if a language L has a 4-round, black-box, computational zero-knowledge proof system with negligible soundness error, then $\bar{L} \in \mathsf{MA}$. Assuming the polynomial hierarchy does not collapse, this means in particular that NP-complete languages do not have 4-round zero-knowledge proofs (at least with respect to black-box simulation).

1 Introduction

A zero-knowledge proof system [23] for a language L is a protocol that enables a prover $\mathcal{P}$ to convince a polynomial-time verifier $\mathcal{V}$ that a given instance x is indeed a member of L. Roughly speaking, the guarantees provided are:

Completeness: If $x \in L$ then the honest prover $\mathcal{P}$ will convince the honest verifier $\mathcal{V}$ to accept, except possibly with some small probability. If $\mathcal{P}$ always convinces $\mathcal{V}$ to accept when $x \in L$ then we say the proof system has *perfect completeness.*

Soundness: If $x \notin L$ a cheating prover $\mathcal{P}^*$ will be unable to falsely convince the honest verifier that x is in L, except with some small probability known as the *soundness error.*

Zero knowledge: When $x \in L$ and the prover is honest, even a malicious verifier $\mathcal{V}^*$ "learns nothing" beyond the fact that $x \in L$.

There are various ways of formalizing the above properties. In this paper, we are interested in the case when the soundness property holds against all-powerful provers — i.e., we focus on *proofs* rather than *arguments* [13] — and we are interested in proof systems with negligible soundness error. For the proof system to be non-trivial, the completeness error should not be too large; we will consider both the case of perfect completeness as well as the case when, for $x \in L$, the honest verifier accepts with some noticeable (i.e., inverse polynomial) probability. Finally, we focus on the case of *computational* zero knowledge (CZK) where, informally, the requirement is that a non-uniform *polynomial-time* cheating verifier learns nothing from the interaction. (Formal definitions are provided

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in Section 2.) We let CZK denote the class of languages that admit a computational zero-knowledge proof system.

In this paper we study the round complexity of CZK proof systems, where a round consists of a message sent from one party to the other and we assume that the prover and the verifier speak in alternating rounds. We briefly survey what is known in this regard:

Unconditional constructions. The only languages currently known to be in CZK *unconditionally* are those that admit *statistical* zero-knowledge (SZK) proofs [23] where, informally, even an all-powerful cheating verifier learns nothing from its interaction with the prover; we denote the class of languages admitting statistical zero-knowledge proofs by SZK. While it is not known[1] whether all languages in SZK have constant-round statistical zero-knowledge proof systems, such proof systems are known for specific languages. In particular, graph non-isomorphism [21] (cf. [21, Remark 12]) as well as languages related to various number-theoretic problems [23, 30, 34, 15, 31, 14] have 4-round SZK proof systems, and graph isomorphism [7] has a 5-round SZK proof system.

Constructions based on one-way functions/permutations. Assuming the existence of one-way functions, every language in NP has an $\omega(1)$-round CZK proof system where the honest prover runs in polynomial time given an NP-witness for the statement being proved [21]. (Actually, this result holds for MA as well.[2]) If no computational restrictions are placed on the honest prover, then any language in AM has an $\omega(1)$-round CZK proof system under the same assumption, and any language in $\mathsf{IP} = \mathsf{PSPACE}$ has a CZK proof system with polynomially-many rounds [29, 10].

Assuming the existence of one-way permutations, Feige and Shamir [17] show a 4-round computational zero-knowledge *argument* for any language in NP. Their techniques yield a 5-round CZK argument based on one-way functions, and this was later improved to 4 rounds by Bellare et al. [6].

Constructions based on stronger assumptions. Assuming the existence of a two-round statistically-hiding commitment scheme, there exists a 5-round CZK proof system for any language in NP [19] (or even AM if the honest prover can be unbounded. (More generally, given a constant-round statistically-hiding commitment scheme, there exists a constant-round CZK proof system for any language in AM.(Two-round statistically-hiding commitment schemes, in turn, can be constructed based on a variety of number-theoretic assumptions [12, 13, 24] or the existence of collision-resistant hash functions [16, 28].

Although statistically-hiding commitment schemes can be constructed from any one-way function [27], constructions of *constant-round* statistically-hiding commitment schemes from one-way functions are unlikely to exist [26].

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[1] The constant-round proofs in [8], based on specific number-theoretic assumptions, consider a weaker variant of SZK where the verifier is assumed to run in polynomial time during its interaction with the prover. See also [35].

[2] The class MA is defined in Section 2. AM denotes the class of languages having *constant-round* Arthur-Merlin proofs.

Lower bounds. Goldreich and Oren [22] show that 2-round CZK proofs exist only for languages in BPP. (Their result applies to *auxiliary-input* zero knowledge proofs, the type we will be concerned with here as well.) Extending this result, Goldreich and Krawczyk [20] show that 3-round *black-box* CZK proofs exist only for languages in BPP. (A definition of black-box CZK is given in Section 2.) Both these results hold for arguments as well as proofs.

Pass [33] gives evidence of the difficulty of showing a black-box *construction* of a constant-round CZK proof for NP based on any one-way function (even if non black-box *simulation* is allowed). We refer to his paper for a precise statement of this result.

1.1 Our Result

We show that 4-round black-box CZK proofs, even with imperfect completeness, exist only for languages whose complement is in MA. This result is unconditional, and holds independent of any cryptographic assumptions one might make. Other than the fact that the bound holds only with respect to black-box simulation, this result is essentially the best one could hope for:

– Under widely-believed number-theoretic assumptions, there exist 5-round CZK proofs for all of NP [19]. Assuming the polynomial hierarchy does not collapse [11], our result indicates that the round complexity in this case is optimal.
– Our result applies only to proofs, but not arguments. Indeed, as noted earlier, there exist 4-round CZK *arguments* for all of NP under relatively weak assumptions [17, 6].
– There exist unconditional constructions of 4-round CZK proofs for languages believed to be outside of BPP, such as graph non-isomorphism [21].

We remark also that for the case of *uniform* zero-knowledge (i.e., protocols which are zero knowledge for *uniform* polynomial-time verifiers), a 4-round protocol for all of NP is possible [19] assuming the existence of 1-round statistically-hiding commitment schemes (that are computationally binding for uniform adversaries).

Besides shedding further light on the finer structure of the class CZK, our result indicates that (black-box) 4-round CZK proofs for all of NP are impossible and so the round complexity of the Goldreich-Kahan protocol [19] is optimal. Our result also gives an "explanation" as to why the known SZK proof for graph isomorphism requires five rounds [7] even though graph *non*-isomorphism has a 4-round SZK proof [21].

Limitations of black-box impossibility results. We prove our result only for the case of *black-box* zero-knowledge protocols. The work of Barak [3], however, shows that black-box impossibility results and lower bounds need not carry over to the general case.[3] Nevertheless, black-box bounds are useful insofar as they

[3] Barak's work gives a constant-round, public-coin, CZK argument for all of NP, something that was ruled out with respect to black-box simulation by Goldreich and Krawczyk [20].

rule out a particular approach for solving a problem. We remark further that many of the known (natural) zero-knowledge *proofs* are in fact black-box zero knowledge; in particular, the protocols of Barak [3] as well as those based on "knowledge of exponent" assumptions [25, 9] are zero-knowledge *arguments*. On the other hand, non black-box zero-knowledge *proofs* using four or fewer rounds are known to exist based on various non-standard assumptions [5, 32].

Our current ability to prove *general* (as opposed to black-box) lower bounds for zero-knowledge protocols is, unfortunately, relatively limited [22, 5].

High-level overview of our technique. Our lower bound for 4-round protocols is proved by extending the Goldreich-Krawczyk lower bound [20] for 3-round protocols. (We assume familiarity with their proof in what follows.) To prove their result, Goldreich and Krawczyk consider a cheating verifier $\mathcal{V}^*$ who generates its message, in the second round of the protocol, using fresh random coins that are determined as a function of the prover's first message. On an intuitive level this means that rewinding is useless because every time V^* is rewound, and a different first message is sent by the simulator, it is as if the protocol execution is being started again from scratch.

We use the same basic idea, now applied to the verifier's message sent in the *third* round of the protocol. A problem is that the verifier's first-round message may "commit" the verifier, in a computational sense, to only one possible third-round message. (Roughly speaking, the verifier cannot be committed in an information-theoretic sense because then an all-powerful prover could guess the third-round message *in advance* based on the first-round message alone. This is one reason why our result applies only to proofs, and not arguments.) For this reason, we use some "all-powerful" entity to provide the verifier with *collisions*, i.e., multiple third-round messages consistent with the same first-round message. This idea was directly inspired by the recent work of Haitner et al. [26], who use such collisions to prove lower bounds on the round complexity of black-box constructions of interactive protocols in other settings. In their work, an oracle provides collisions. Here, we do not have an oracle; instead, we have an all-powerful prover that provides collisions as part of an interactive MA-proof for some language. See Section 3 for further intuition, as well as the details of the proof.

1.2 Outline of the Paper

Standard definitions, as well as some terminology specific to this paper, are provided in Section 2. In Section 3 we prove our result for the case of CZK proof systems with perfect completeness. Technical modifications necessary to deal with the case of imperfect completeness are deferred to Section 4. We conclude with some open questions in Section 5.

2 Definitions

Given interactive algorithms $\mathcal{P}$ and $\mathcal{V}$, we let $\langle \mathcal{P}(x), \mathcal{V}(y) \rangle$ denote the interaction of $\mathcal{P}$, holding input x, with $\mathcal{V}$, holding input y. We let $\langle \mathcal{P}(x), \mathcal{V}(y) \rangle = 1$ denote the event that $\mathcal{V}$ outputs 1 in the indicated interaction, where an output of "1" is interpreted as "accept" and an output of "0" is interpreted as "reject". We now give the standard definition of an interactive proof system [23] for a language L.

Definition 1. *Interactive algorithms $\mathcal{P}, \mathcal{V}$ form an* interactive proof system *for a language L if $\mathcal{V}$ runs in probabilistic polynomial time and there exist non-negative functions c, s such that:*

- *For all $x \in L$, it holds that $\Pr[\langle \mathcal{P}(x), \mathcal{V}(x) \rangle = 1] \geq c(|x|)$. (Note that we do not require $\mathcal{P}$ to run in polynomial time.)*
- *For all $x \notin L$ and any $\mathcal{P}^*$ we have $\Pr[\langle \mathcal{P}^*, \mathcal{V}(x) \rangle = 1] \leq s(|x|)$.*
- *There exists a polynomial p such that $c(|x|) \geq s(|x|) + 1/p(|x|)$.*

We call c the acceptance probability*, and s the* soundness error*. If $c(|x|) = 1$ for all x, we say the proof system has* perfect completeness*. If s is negligible, we say the proof system has* negligible soundness error*.* ♢

We will only consider zero-knowledge proof systems having negligible soundness error.

A *round* of an interactive proof system consists of a message sent from one party to the other, and we assume that the prover and the verifier speak in alternating rounds. Following [2], we let MA denote the class of languages having a 1-round proof system and in this case refer to the prover as *Merlin* and the verifier as *Arthur*; that is:

Definition 2. *$L \in \mathsf{MA}$ if there exists a probabilistic polynomial-time verifier $\mathcal{V}$, a non-negative function s, and a polynomial p such that the following hold for all sufficiently-long x:*

- *If $x \in L$ then there exists a string w (that can be sent by Merlin) such that*

$$\Pr[\mathcal{V}(x, w) = 1] \geq s(|x|) + 1/p(|x|).$$

- *If $x \notin L$ then for all w (sent by a cheating Merlin) it holds that*

$$\Pr[\mathcal{V}(x, w) = 1] \leq s(|x|).$$

♢

In fact, it is known that an equivalent definition is obtained even if we require perfect completeness and negligible soundness error.

2.1 Zero Knowledge Proof Systems

A *distribution ensemble* $\{X(a)\}_{a \in \{0,1\}^*}$ is an infinite sequence of probability distributions, where a distribution $X(a)$ is associated with each value of a. Two distribution ensembles X and Y are *computationally indistinguishable* if for all polynomial-time algorithms D, there exists a negligible function μ such that for every a we have

$$\left|\Pr[D(X(a), a) = 1] - \Pr[D(Y(a), a) = 1]\right| \leq \mu(|a|).$$

(We do not need to consider non-uniform distinguishers here since non-uniformity can be incorporated via the auxiliary input that we will provide to the cheating verifier, below.)

Given interactive algorithms $\mathcal{P}, \mathcal{V}^*$, we let $\mathsf{trans}_{\mathcal{V}^*}\langle \mathcal{P}(x), \mathcal{V}^*(y)\rangle$ denote the transcript of the indicated interaction; for convenience, this includes both messages of the prover as well as those of the verifier. (We remark that we do not need to consider the entire *view* of $\mathcal{V}^*$ since we will restrict to deterministic verifiers, as justified below, and the input y of $\mathcal{V}^*$ will be provided to the distinguisher as per our definition of computational indistinguishability, above.) We now review the standard definitions for computational zero-knowledge proofs.

Definition 3. *An interactive proof system $\mathcal{P}, \mathcal{V}$ for a language L is said to be a* computational zero-knowledge *proof system if for any probabilistic polynomial-time algorithm $\mathcal{V}^*$, there exists an expected polynomial-time simulator $\mathcal{S}$ such that the following distribution ensembles are computationally indistinguishable:*

$$\left\{\mathsf{trans}_{\mathcal{V}^*}\langle \mathcal{P}(x), \mathcal{V}^*(x,z)\rangle\right\}_{x\in L, z\in\{0,1\}^*} \quad \textit{and} \quad \left\{\mathcal{S}(x,z)\right\}_{x\in L, z\in\{0,1\}^*} .$$

$\diamondsuit$

The above definition incorporates an auxiliary input z provided to $\mathcal{V}^*$, and we may therefore restrict our consideration to verifiers $\mathcal{V}^*$ that are deterministic. Note also that we allow simulation in *expected* polynomial time; this makes our results stronger. (Also, constant-round, black-box CZK proofs with strict polynomial-time simulation are already ruled out by Barak and Lindell [4].)

A computational zero-knowledge proof system $(\mathcal{P}, \mathcal{V})$ is *black-box* zero knowledge if there exists a "universal" simulator that takes oracle access to the cheating verifier $\mathcal{V}^*$. That is:

Definition 4. *A computational zero-knowledge proof system $\mathcal{P}, \mathcal{V}$ is* black-box zero-knowledge *if there exists an expected polynomial-time oracle machine* Sim *(the* black-box simulator*) such that for any probabilistic polynomial-time algorithm $\mathcal{V}^*$ the following distribution ensembles are computationally indistinguishable:*

$$\left\{\mathsf{trans}_{\mathcal{V}^*}\langle \mathcal{P}(x), \mathcal{V}^*(x,z)\rangle\right\}_{x\in L, z\in\{0,1\}^*} \quad \textit{and} \quad \left\{\mathsf{Sim}^{\mathcal{V}^*(x,z)}(x)\right\}_{x\in L, z\in\{0,1\}^*} .$$

$\diamondsuit$

We denote by ${}^{bb}\mathsf{CZK}(r)$ the class of languages that have r-round, *black-box*, computational zero-knowledge proof systems with negligible soundness error.

Terminology and simplifying assumptions. We will be concerned with 4-round CZK proof systems, where (without loss of generality) the verifier sends the first message and the prover sends the final message. We use $\alpha, \beta, \gamma, \delta$ to denote the first, second, third, and fourth messages, respectively. We let $\mathcal{P}_x$ (resp., $\mathcal{V}_x$) denote the honest prover (resp., honest verifier) algorithm when the common input is x.

We let $\alpha = \mathcal{V}_x(r)$ denote the first message sent by $\mathcal{V}_x$ when its random coins are fixed to r, and let $\gamma = \mathcal{V}_x(\alpha, \beta; r)$ denote the third message sent by $\mathcal{V}_x$ in this case. Finally, $\mathcal{V}_x(\alpha, \beta, \gamma, \delta; r)$ is a bit denoting whether the verifier accepts (i.e., outputs 1) or rejects. We say that $(\alpha, \beta, \gamma, \delta, r)$ is an *accepting transcript* for a given x if $\mathcal{V}_x(\alpha, \beta, \gamma, \delta; r) = 1$. Note that we do not require the verifier's decision to depend on the actual transcript alone, but allow its decision to also possibly depend on its random coins.

Without loss of generality, we make a number of simplifying assumptions about the behavior of black-box simulator Sim. The first query of Sim to $\mathcal{V}^*$ will simply be a "prompt" query to which $\mathcal{V}^*$ responds with α. Subsequent queries by Sim are all of the form (α, β) (for some β of Sim's choice), to which $\mathcal{V}^*$ will respond with some γ. (We can assume Sim makes no queries of the form $(\alpha, \beta, \gamma, \delta)$ since $\mathcal{V}^*$ can simply refuse to respond to such queries.) We assume Sim makes a given query only once. Finally, if the simulator outputs the transcript $(\alpha', \beta, \gamma, \delta)$ we assume that $\alpha' = \alpha$, and that the simulator previously queried (α, β) to $\mathcal{V}^*$ and received response γ.

3 CZK Proof Systems with Perfect Completeness

We now state our main result:

Theorem 1. ${}^{bb}\mathsf{CZK}(4) \subseteq \mathsf{coMA}$.

In this section we prove this result in the easier case when the proof system in question has perfect completeness; we handle the case of imperfect completeness in the following section.

As intuition for the proof, consider first the case of a malicious verifier $\hat{\mathcal{V}}$ who acts in the following way: it sends an initial message α, and then in response to the prover's second message β it chooses a random message γ consistent with α. (For now, we do not worry about the fact that this does not necessarily represent a feasible polynomial-time strategy.) Formally, if we let R_α denote the set of random coins consistent with α (i.e., $r \in R_\alpha$ implies $\mathcal{V}_x(r) = \alpha$), then in response to β the malicious verifier chooses a random $r \in R_\alpha$ and computes $\gamma = \mathcal{V}_x(\alpha, \beta; r)$. Intuitively, it will be difficult to simulate an accepting transcript for such a verifier since each time the simulator "rewinds" $\hat{\mathcal{V}}$ it will be given a message γ consistent with a *different* set of random coins. In fact, we can prove that if $x \notin L$ then the simulator will *not* be able to simulate an accepting transcript for such a verifier, since the ability to do so with non-negligible probability could be translated into the ability to violate the soundness condition of the proof system with non-negligible probability. (A proof of this fact goes along similar lines as the proof in [20].)

On the other hand, consider the case when $x \in L$. From the perspective of the honest prover, the behavior of $\hat{\mathcal{V}}$ is identical to that of the honest verifier, and so the honest prover's interaction with $\hat{\mathcal{V}}$ leads to an accepting transcript with probability 1. We would like to claim that the zero-knowledge condition implies that Sim simulates an accepting transcript for such a verifier with high probability. Unfortunately, $\hat{\mathcal{V}}$ as described above may not run in polynomial time, whereas simulation is only guaranteed for polynomial-time verifiers.

It is possible, however, to obtain a polynomial-time cheating verifier with the desired behavior by providing the verifier as *auxiliary input* a sequence of sufficiently-many coins $r_1, \ldots, r_s$ that are all consistent with the same first message α. Specifically, consider the verifier $\mathcal{V}^*$ defined as follows: given auxiliary input $r_1, \ldots, r_s$ (all consistent with the same first message α) and a poly-wise independent hash function h, send α as the first message. In response to the prover's second message β, compute $i = h(\beta)$ and use r_i to compute the next message $\gamma = \mathcal{V}_x(\alpha, \beta; r_i)$. Note that if $r_1, \ldots, r_s$ are chosen at random (subject to the constraint that they are all mutually consistent) then the behavior of $\mathcal{V}^*$ is identical to the behavior of $\hat{\mathcal{V}}$ as far as the honest prover is concerned. Since $\mathcal{V}^*$ runs in polynomial time, we are now able to argue that Sim simulates an accepting transcript for $\mathcal{V}^*$ with high probability when $x \in L$. Furthermore, it is still possible to show (using a slightly more complicated argument) that, with overwhelming probability, Sim *fails* to simulate an accepting transcript for this verifier whenever $x \notin L$.

Based on the above, we obtain an MA proof system for $\bar{L}$: Merlin sends Arthur a sequence $r_1, \ldots, r_s$ of random coins that are all consistent with the same first message α, and Arthur simulates an execution of $\mathsf{Sim}^{\mathcal{V}^*}(x)$. If this does *not* result in an accepting transcript then Arthur accepts, while if it does lead to an accepting transcript then Arthur rejects.

We now formalize the above intuition and show how to handle various technicalities that arise. Fix $L \in {}^{bb}\mathsf{CZK}(4)$. This means that, for this language, there exists a prover $\mathcal{P}$, a verifier $\mathcal{V}$, and a black-box simulator Sim satisfying Definitions 1–4 (except that, in this section, we are assuming perfect completeness). Assume without loss of generality that the second message of the protocol always has length $m(\cdot)$, and let $\ell(\cdot)$ denote the number of random coins used by $\mathcal{V}$. Let $T(\cdot)$ denote an upper-bound on the expected running time of Sim.

Consider the following MA proof system for the language $\bar{L}$, where Merlin (i.e., the prover) and Arthur (i.e., the verifier) share in advance an input x of length n:

Notation: Let $\ell = \ell(n)$, $m = m(n)$, and $T = T(n)$. Set $s = 50 \cdot T^2$; note that s is polynomial in n.

Merlin's message: Merlin sends a sequence of s coins $r_1, \ldots, r_s \in \{0,1\}^\ell$. (For the honest Merlin, these are all consistent with the same first message α.)

Arthur's actions: Arthur proceeds as follows:

1. Set $\alpha = \mathcal{V}_x(r_1)$. Check that $\alpha = \mathcal{V}_x(r_i)$ for all $1 < i \leq s$, i.e., that all the random coins are consistent with the same first message α. If not, reject; otherwise, go to the next step.
2. Choose a random $5T$-wise independent hash function $h : \{0,1\}^m \to \{1, \ldots, s\}$. Construct the following deterministic verifier $\mathcal{V}^*$:
   (a) Send first message α to the prover.

(b) Upon receiving message β from the prover, compute $i = h(\beta)$ and send the message $\gamma = \mathcal{V}_x(\alpha, \beta; r_i)$ to the prover.

3. Run $\mathsf{Sim}^{\mathcal{V}^*}(x)$ for at most $5T$ steps using uniformly-chosen random coins for Sim. If Sim does *not* output an accepting transcript within this time bound, output "accept". Otherwise, output "reject". (Formally, output "reject" iff Sim outputs $(\alpha, \beta, \gamma, \delta)$, within the allotted time bound, such that $\mathcal{V}_x(\alpha, \beta, \gamma, \delta; r_{h(\beta)}) = 1$.)

The following claims show that the above is a valid MA-protocol for $\bar{L}$, thus proving Theorem 1 for the case of protocols having perfect completeness.

Claim 1. *For any $x \notin \bar{L}$ sufficiently long and for any message $r_1, \ldots, r_s$ sent by Merlin, the probability that Arthur accepts is at most $2/5$.*

Proof. Fix some $r_1, \ldots, r_s$ sent by Merlin. Assume $\mathcal{V}_x(r_i) = \mathcal{V}_x(r_j)$ for all $1 \leq i, j \leq s$ since, if not, Arthur rejects immediately. When $x \notin \bar{L}$ we have $x \in L$ and, by perfect completeness, the interaction of the honest prover $\mathcal{P}_x$ with $\mathcal{V}^*$ would result in an accepting transcript with probability 1. (To see this, note that an execution of $\mathcal{V}^*$ is equivalent to an execution of the honest verifier $\mathcal{V}_x$ using random coins $r_{h(\beta)}$.) The zero-knowledge condition thus implies that, for x sufficiently long, $\mathsf{Sim}^{\mathcal{V}^*}(x)$ outputs an accepting conversation with probability at least $4/5$. It follows that even the truncated version of Sim, where its execution is halted after $5T$ steps, outputs an accepting conversation with probability at least $3/5$. Arthur thus accepts with probability at most $2/5$, as claimed.

Claim 2. *For any $x \in \bar{L}$ sufficiently long, there exists a message $r_1, \ldots, r_s$ such that Arthur will accept with probability at least $1/2$.*

Proof. Fix $x \in \bar{L}$. We show a randomized strategy that allows Merlin to convince Arthur with probability at least $1/2$; this implies the claim.

Merlin proceeds as follows: choose random $r_1 \in \{0,1\}^\ell$ and compute $\alpha = \mathcal{V}_x(r_1)$. Define $R_\alpha \stackrel{\text{def}}{=} \{r \mid \mathcal{V}_x(r) = \alpha\}$; i.e., R_α is the set of coins for the honest verifier consistent with the first message α. Then choose $r_2, \ldots, r_s$ uniformly from R_α. (These need not be distinct.) Send $r_1, \ldots, r_s$ to Arthur. Let p^* denote the probability that Arthur rejects. Note that this is exactly the probability that $\mathsf{Sim}^{\mathcal{V}^*}(x)$ outputs an accepting transcript within the allotted time bound.

We upper-bound p^* by considering a slightly different experiment involving an all-powerful cheating prover $\mathcal{P}^*$ attempting to falsely convince the honest verifier $\mathcal{V}_x$ that $x \in L$. The strategy of $\mathcal{P}^*$ is defined as follows:

1. Receive message α from the verifier. Let $R_\alpha \stackrel{\text{def}}{=} \{r \mid \mathcal{V}_x(r) = \alpha\}$.
2. Run Sim using uniformly-chosen random coins, for at most $5T$ steps. Sim expects to be given oracle access to a (cheating) verifier, and $\mathcal{P}^*$ simulates the actions of such a verifier as follows:
   (a) Choose a random index $q \leftarrow \{1, \ldots, 5T\}$.
   (b) Send α as the verifier's first message.

(c) In response to the i^{th} simulator message (α, β_i) for $i \neq q$, choose a random $r_i \leftarrow R_\alpha$, compute $\gamma_i = \mathcal{V}_x(\alpha, \beta_i; r_i)$, and give γ_i to Sim. (Recall we assume that Sim never makes the same query twice.)

(d) In response to the q^{th} simulator message (α, β_q), send β_q to the (external) honest verifier, and receive in return a message γ_q. Give γ_q to Sim.

3. If Sim outputs a conversation $(\alpha, \beta, \gamma, \delta)$ with $\beta = \beta_q$ within the allotted time bound, then send δ to the (external) honest verifier.

In the above experiment, each "query" β_i of Sim is answered by using a random element $r_i \leftarrow R_\alpha$ to compute the response $\gamma_i = \mathcal{V}_x(\alpha, \beta_i; r_i)$. This is immediate for $i \neq q$, but is true also for $i = q$ since, from the perspective of $\mathcal{P}^*$ and Sim, the coins being used by the external, honest verifier are uniformly-distributed in R_α. Let $\hat{p}$ denote the probability that Sim outputs an accepting transcript in this case, within the allotted time bound. Since Sim makes at most $5T$ queries to its oracle in the above experiment, $\mathcal{P}^*$ convinces the honest verifier to accept with probability $\hat{p}/5T$. Since the proof system has negligible soundness error we have that, for x sufficiently long, $\hat{p} \leq 1/4$.

We return now to consideration of p^*. When Arthur runs $\mathsf{Sim}^{\mathcal{V}^*}(x)$, he does so by first choosing a random h and then answering the simulator's i^{th} query (α, β_i) by using $r_{h(\beta_i)}$ to compute the response $\gamma_i = \mathcal{V}_x(\alpha, \beta_i; r_{h(\beta_i)})$. Since Merlin chooses each of the r_i uniformly from R_α, these responses are distributed identically to the above experiment unless there is a collision in h; that is, unless there exist some $\beta_i \neq \beta_j$ with $h(\beta_i) = h(\beta_j)$. Because h is chosen in a $5T$-wise independent fashion and Sim is restricted to making only $5T$ queries, a standard birthday bound shows that the probability of such a collision is at most $(5T)^2/2s = 1/4$. Conditioned on a collision not occurring, the probability that $\mathsf{Sim}^{\mathcal{V}^*}(x)$ outputs an accepting conversation is exactly $\hat{p} \leq 1/4$. We conclude that $p^* \leq 1/4 + 1/4 = 1/2$, and so Arthur rejects with probability at most $1/2$ (and accepts with probability at least $1/2$).

4 Handling Imperfect Completeness

In the previous section we assumed perfect completeness, and in fact this is essential for the MA proof system given there. To see the problem, assume $\mathcal{P}, \mathcal{V}$ is such that the honest verifier immediately rejects whenever its random coins are all 0. Then a cheating Merlin can send $r_1 = \cdots = r_s = 0^\ell$ and this will cause Arthur to accept with probability 1 even when $x \notin \bar{L}$.

In the modified proof system, we have Arthur "verify" that Merlin sends "representative" random coins $r_1, \ldots, r_s$ by checking that $\mathsf{Sim}^{\mathcal{V}_x(r_i)}(x)$, for a random element r_i in the set sent by Merlin, outputs an accepting transcript with "high" probability. Then Arthur checks whether $\mathsf{Sim}^{\mathcal{V}^*(x; r_1, \ldots, r_s, h)}(x)$ *fails* to output an accepting transcript, as in the previous section. Unfortunately, this may make the honest Merlin's job harder when $x \in \bar{L}$ since in this case $\mathsf{Sim}^{\mathcal{V}_x(r_i)}(x)$ might (legitimately) *never* output an accepting transcript. But Arthur can easily check for this by running $\mathsf{Sim}^{\mathcal{V}_x(r)}(x)$ using random coins r that it chooses itself.

We remark that if we were content to show inclusion in AM (rather than MA), the proof could be simplified somewhat.

Before presenting the modified proof system, we introduce some notation. For a given randomized experiment Expt that can be run in polynomial time, we let $\mathsf{estimate}_\varepsilon(\Pr_r[\mathsf{Expt}])$ denote a procedure that outputs an estimate to the given probability (taken over randomness r) to within an additive factor of ε, except with probability at most ε. That is:

$$\Pr\left[\,\left|\mathsf{estimate}_\varepsilon(\Pr_r[\mathsf{Expt}=1]) - \Pr_r[\mathsf{Expt}=1]\right| \geq \varepsilon\right] \leq \varepsilon.$$

This can be done in the standard way using $\Theta(\varepsilon^{-2}\log\frac{1}{\varepsilon})$ independent executions of Expt. The important thing to note is that when ε is noticeable, this estimation can be done in polynomial time. In the experiments we will be considering, some variables will be fixed as part of the experiment and others will be chosen at random; we will always subscript those variables being chosen at random (as done above with the subscripted r).

Below, we let $\mathcal{V}^*$ denote the same malicious verifier as in the previous section. Specifically, on input x and auxiliary input $z = r_1, \ldots, r_s, h$, where each r_i represents coins for the honest verifier and h is a hash function, $\mathcal{V}^*$ acts as follows:

1. Send first message $\alpha = \mathcal{V}_x(r_1)$ to the prover.
2. Upon receiving message β from the prover, compute $i = h(\beta)$ and send the message $\gamma = \mathcal{V}_x(\alpha, \beta; r_i)$ to the prover.
3. Receive final message δ from the prover.

We say an interaction of $\mathcal{P}_x$ with $\mathcal{V}^*(x,z)$ *results in an accepting transcript* if $(\alpha, \beta, \gamma, \delta, r_i)$ is an accepting transcript.

Let $L \in {}^{bb}\mathsf{CZK}(4)$, and assume L has a 4-round CZK proof system $\mathcal{P}, \mathcal{V}$ with acceptance probability $c(\cdot)$ where c is noticeable (i.e., $c = \Omega(1/p)$ for some polynomial p). Let ℓ, m, and T be as in the previous section. Once again, Merlin and Arthur share in advance an input x of length n. The MA proof system for the language $\bar{L}$ follows:

Notation: Let $c = c(n)$, $\ell = \ell(n)$, $m = m(n)$, and $T = T(n)$. Assume n is large enough so that $c > 0$. Set $\varepsilon = c/20$, and $s = 4T^2\varepsilon^{-3}$. (Note that ε is noticeable, and s is polynomial.) Let $\widetilde{\mathsf{Sim}}$ denote an execution of Sim for at most $2T/\varepsilon$ steps.

Merlin's message: Merlin sends a sequence of s coins $r_1, \ldots, r_s \in \{0,1\}^\ell$.

Arthur's actions: Arthur proceeds as follows:

1. Compute

$$p_1 = \mathsf{estimate}_\varepsilon\left(\Pr_{r',r}\left[\widetilde{\mathsf{Sim}}^{\mathcal{V}_x(r')}(x;r) \text{ outputs an accepting transcript}\right]\right).$$

   If $p_1 < c - 2\varepsilon$ then accept; otherwise, continue to the next step.
2. Set $\alpha = \mathcal{V}_x(r_1)$. Check that $\alpha = \mathcal{V}_x(r_i)$ for all $1 < i \leq s$. If not, reject; otherwise, continue to the next step.

3. Choose $i \leftarrow \{1, \ldots, s\}$ and coins r and run $\widetilde{\mathsf{Sim}}^{\mathcal{V}_x(r_i)}(x; r)$. If this does not result in an accepting transcript, reject; otherwise, continue to the next step.
4. Let H denote a family of $2T/\varepsilon$-wise independent hash functions $h : \{0,1\}^m \to \{1, \ldots, s\}$. Compute

   $$p_2 = \\ \mathsf{estimate}_\varepsilon \left(\Pr_{h \leftarrow H, r} \left[\widetilde{\mathsf{Sim}}^{\mathcal{V}^*(x; \boldsymbol{r}, h)}(x; r) \text{ outputs an accepting transcript} \right] \right),$$

   where $\boldsymbol{r} = (r_1, \ldots, r_s)$. If $p_2 < c - 10\varepsilon$ accept; else reject.

(It should be clear that we have not attempted to optimize any of the parameters of the above proof system.) We now prove claims analogous to those in the previous section.

Claim 3. *For any $x \notin \bar{L}$ sufficiently long and for any message $r_1, \ldots, r_s$ sent by Merlin, the probability that Arthur accepts is at most $c - 6\varepsilon$.*

Proof. If $x \notin \bar{L}$ then $x \in L$ and so the interaction of $\mathcal{P}_x$ with $\mathcal{V}_x$ results in an accepting transcript with probability at least c. The zero-knowledge condition implies that, for x sufficiently long,

$$\Pr_{r', r}[\widetilde{\mathsf{Sim}}^{\mathcal{V}_x(r')}(x; r) \text{ outputs an accepting transcript}] \geq c - \varepsilon.$$

This means that, except with probability at most ε, the value p_1 computed by Arthur satisfies $p_1 \geq c - 2\varepsilon$; thus, Arthur accepts in the first step with probability at most ε.

Fix some $\boldsymbol{r} = (r_1, \ldots, r_s)$ sent by Merlin. We may assume $\mathcal{V}_x(r_i) = \mathcal{V}_x(r_j)$ for all $1 \leq i, j \leq s$ since, if not, Arthur rejects in the second step. Define

$$\hat{p} = \Pr_{i \leftarrow \{1, \ldots, s\}, r} \left[\widetilde{\mathsf{Sim}}^{\mathcal{V}_x(r_i)}(x; r) \text{ outputs an accepting transcript} \right].$$

There are two cases to consider:

Case 1: If $\hat{p} < c - 7\varepsilon$, then the probability that Arthur does not reject in step 3 is at most $c - 7\varepsilon$.

Case 2: On the other hand, if $\hat{p} \geq c - 7\varepsilon$ then (again using the zero-knowledge property)

$$\Pr_{i \leftarrow \{1, \ldots, s\}, r} \left[\langle \mathcal{P}_x(r), \mathcal{V}_x(r_i) \rangle = 1 \right] \geq c - 8\varepsilon.$$

By definition of $\mathcal{V}^*$ it holds that

$$\Pr_{h \leftarrow H, r} \left[\langle \mathcal{P}_x(r), \mathcal{V}^*(x, \boldsymbol{r}, h) \rangle \text{ results in an accepting transcript} \right] \\ = \Pr_{i \leftarrow \{1, \ldots, s\}, r} \left[\langle \mathcal{P}_x(r), \mathcal{V}_x(r_i) \rangle = 1 \right].$$

Thus, relying on the zero-knowledge property once again,

$$\Pr_{h \leftarrow H, r} \left[\widetilde{\mathsf{Sim}}^{\mathcal{V}^*(x; r_1, \ldots, r_s, h)}(x; r) \text{ outputs an accepting transcript} \right] \geq c - 9\varepsilon.$$

So, except with probability at most ε, the value p_2 computed by Arthur satisfies $p_2 \geq c - 10\varepsilon$; thus, Arthur accepts in the last step with probability at most ε.

Combining the above, we see that Arthur accepts with probability at most $\varepsilon + \max\{c - 7\varepsilon, \varepsilon\}$, which is at most $c - 6\varepsilon$.

Claim 4. *For any $x \in \bar{L}$ sufficiently long, there exists a message $r_1, \ldots, r_s$ such that Arthur will accept with probability at least $c - 5\varepsilon$.*

Proof. Fix $x \in \bar{L}$. Define

$$\hat{p} = \Pr_{r',r}\left[\widetilde{\mathsf{Sim}}^{\mathcal{V}_x(r')}(x; r) \text{ outputs an accepting transcript}\right].$$

There are two cases to consider:

Case 1: If $\hat{p} < c - 3\varepsilon$ then, except with probability at most ε, the value p_1 computed by Arthur satisfies $p_1 < c - 2\varepsilon$; thus, Arthur accepts in the first step with probability at least $1 - \varepsilon \geq c - 5\varepsilon$.

Case 2: On the other hand, say $\hat{p} \geq c - 3\varepsilon$. As in the proof of Claim 2, Merlin proceeds as follows: choose random $r_1 \in \{0,1\}^\ell$ and compute $\alpha = \mathcal{V}_x(r_1)$. Let $R_\alpha \stackrel{\text{def}}{=} \{r \mid \mathcal{V}_x(r) = \alpha\}$, and choose $r_2, \ldots, r_s$ uniformly from R_α. Send $\boldsymbol{r} = (r_1, \ldots, r_s)$ to Arthur. We show that Arthur will accept with high probability.

Arthur can reject in either step 3 or step 4. We upper-bound the probability that Arthur rejects in either of these steps individually, and then apply a union bound to upper-bound the total probability that Arthur rejects.

Each r_i, taken individually, is uniformly distributed in $\{0,1\}^\ell$. Thus, in step 3, choosing a random $i \in \{1, \ldots, s\}$ and using coins r_i is equivalent to choosing uniformly-random coins for $\mathcal{V}_x$. It follows that the probability that Arthur rejects in step 3 is exactly equal to $1 - \hat{p} \leq 1 - c + 3\varepsilon$.

We proceed to analyze step 4. As in the proof of Claim 2, say a *collision* occurs in an execution of $\widetilde{\mathsf{Sim}}^{\mathcal{V}^*(x;r_1,\ldots,r_s,h)}(x; r)$ if the simulator makes two distinct queries (α, β_i) and (α, β_j) for which $h(\beta_i) = h(\beta_j)$. Let coll denote such an event. As before, we have

$$\Pr_{\boldsymbol{r},h,r}\left[\widetilde{\mathsf{Sim}}^{\mathcal{V}^*(x;\boldsymbol{r},h)}(x; r) \text{ outputs an accepting transcript}\right] \leq \tag{1}$$
$$\Pr_{\boldsymbol{r},h,r}[\mathsf{coll}] + \Pr_{\boldsymbol{r},h,r}\left[\widetilde{\mathsf{Sim}}^{\mathcal{V}^*(x;\boldsymbol{r},h)}(x; r) \text{ outputs an accepting transcript} \mid \overline{\mathsf{coll}}\right],$$

where $\boldsymbol{r} = (r_1, \ldots, r_s)$ are chosen by Merlin as described above (and not uniformly and independently at random). The probability of a collision is independent of $r_1, \ldots, r_s$, and is upper-bounded by $\Pr[\mathsf{coll}] \leq \frac{(2T/\varepsilon)^2}{2s} = \frac{\varepsilon}{2}$. As in the proof of Claim 2, for sufficiently-long x it holds that

$$\Pr_{\boldsymbol{r},h,r}\left[\widetilde{\mathsf{Sim}}^{\mathcal{V}^*(x;\boldsymbol{r},h)}(x; r) \text{ outputs an accepting transcript} \mid \overline{\mathsf{coll}}\right] \leq \varepsilon^2/2;$$

this means that, except with probability at most ε, the $r_1, \ldots, r_s$ chosen by Merlin satisfy

$$\Pr_{h,r}\left[\widetilde{\mathsf{Sim}}^{\mathcal{V}^*(x;r_1,\ldots,r_s,h)}(x;r) \text{ outputs an accepting transcript} \mid \overline{\mathsf{coll}}\right] \leq \varepsilon/2.$$

Using Equation (1), we see that except with probability at most ε, the $r_1, \ldots, r_s$ chosen by Merlin satisfy

$$\Pr_{h,r}\left[\widetilde{\mathsf{Sim}}^{\mathcal{V}^*(x;r_1,\ldots,r_s,h)}(x;r) \text{ outputs an accepting transcript}\right] \leq \varepsilon < c - 11\varepsilon.$$

Assuming the above to be the case, Arthur will reject in step 4 with probability at most ε. Taken together, this means that Arthur rejects in step 4 with probability at most 2ε.

Summing the probabilities of rejection in steps 3 and 4, we see that, overall, Arthur rejects with probability at most $1 - c + 5\varepsilon$, or accepts with probability at least $c - 5\varepsilon$.

5 Future Directions

Coupled with the obvious fact that ${}^{bb}\mathsf{CZK}(4) \subseteq \mathsf{AM}$, this work shows that ${}^{bb}\mathsf{CZK}(4) \subseteq \mathsf{AM} \cap \mathsf{coMA}$. Due to the similarity with the fact that $\mathsf{SZK} \subseteq \mathsf{AM} \cap \mathsf{coAM}$ [18, 1], as well as the fact that the only languages known to be in ${}^{4}\mathsf{CZK}$ (under any assumption) are also in SZK, it is natural to conjecture that ${}^{bb}\mathsf{CZK}(4) \subseteq \mathsf{SZK}$.

Another interesting direction would be to show any broad positive results for ${}^{4}\mathsf{CZK}$: say, along the lines of proving that $\mathsf{NP} \cap \mathsf{coNP} \subseteq {}^{4}\mathsf{CZK}$.

In a slightly different direction, suggested by Hoeteck Wee: can a tighter bound be shown for languages L having 4-round zero-knowledge proofs *of knowledge* (beyond the fact that $L \in \mathsf{NP}$)?

Finally, is it possible to apply the techniques from [26] to show that there are no black-box constructions of *constant-round* (black-box) zero-knowledge proofs for NP?

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